UCSD ECE 153 Handout #4
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Solutions to ECE 153 Aptitude Test

Simplify the following equations as much as you can. Throughout $r \in (0,1)$.

1. $\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

   Solution: Let
   $$S = 1 + r + r^2 + \ldots.$$
   Then
   $$rS = r + r^2 + r^3 + \ldots.$$
   Taking the difference between two equations, we have $S - rS = 1$, or equivalently,
   $$S = \frac{1}{1-r}.$$

2. $\displaystyle\sum_{n=0}^{\infty} nr^n = \frac{r}{(1-r)^2}$.

   Solution: There are two ways to evaluate this.

   Method 1: Write
   $$S = r + 2r^2 + 3r^3 + \ldots.$$
   Then
   $$rS = r^2 + 2r^3 + 3r^4 + \ldots.$$
   By taking the difference between S and rS, we can easily see that
   $$(1-r)S = r + r^2 + r^3 + \ldots = \frac{r}{1-r}.$$

Method 2: We have

$$\begin{aligned}\sum_{n=0}^{\infty} nr^n &= r\sum_{n=0}^{\infty} nr^{n-1}\\ &= r\sum_{n=0}^{\infty} \frac{d}{dr}\left(r^n\right)\\ &= r\frac{d}{dr}\left(\sum_{n=0}^{\infty} r^n\right)\\ &= r\frac{d}{dr}\left(\frac{1}{1-r}\right)\\ &= \frac{r}{(1-r)^2}.\end{aligned}$$

(Note: The interchange of the differentiation and the summation should and can be justified. We already learned a similar and more general method in ECE 109 with moment generating functions; the given identity is nothing but the mean of a geometric random variable.)

3. $\displaystyle\sum_{n=0}^{\infty} \frac{r^n}{n!} = e^r.$

Solution: This is one of famous Maclaurin series (a special form of Taylor series evaluated at 0). The Maclaurin series generated by a function $f(r)$ is

$$f(r) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} r^n$$

where $f^{(n)}(0)$ denotes the nth derivative of f evaluated at 0.

Let $f(r) = e^r$. Since $f^{(n)}(r) = e^r$ for all nonnegative integer n, $f^{(n)}(0) = e^0 = 1$ for all n. Therefore,

$$e^r = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} r^n = \sum_{n=0}^{\infty} \frac{r^n}{n!}.$$

Alternative Solution: The probability mass function of a Poisson random variable X with mean r is given by

$$\mathrm{P}\{X = n\} = e^{-r}\frac{r^n}{n!}$$

which sums to 1, as

$$e^{-r}\sum_{n=0}^{\infty} \frac{r^n}{n!} = 1.$$

By multiplying e^r to both sides, we get

$$\sum_{n=0}^{\infty} \frac{r^n}{n!} = e^r.$$

4. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} = 1$

   Solution: This is just 1 since each term of the summation is the probability of k successes out of n independent trials, each with success probability r. In other words, each term is the probability mass function of a binomial random variable with parameter r which should sum to 1.

   Alternatively, from the binomial expansion of $(q+p)^n$ which is

   $$(q+p)^n = \sum_{k=0}^{n} \binom{n}{k} q^k p^{n-k}.$$

   Taking $q = r$ and $p = 1-r$, we get

   $$1 = (r+1-r)^n = \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k}.$$

5. $\displaystyle\int_0^{\infty} e^{-x} dx = 1.$

   Solution: We have

   $$\int_0^{\infty} e^{-x} dx = -e^{-x}|_0^{\infty} = 0 - (-1) = 1.$$

6. $\displaystyle\int_0^{\infty} x e^{-x} dx = 1.$

   Solution: Integration by parts. Since

   $$(xe^{-x})' = e^{-x} - xe^{-x},$$

   we have

   $$0 = xe^{-x}|_0^{\infty} = \int e^{-x} - \int xe^{-x} = 1 - \int xe^{-x}.$$

7. $\displaystyle\int_0^\infty \int_x^\infty e^{-y} dy dx = 1.$

Solution: There are two ways to evaluate this.

Method 1:

$$\begin{aligned}\int_0^\infty \int_x^\infty e^{-y} dy dx &= \int_0^\infty \left(-e^{-y}\big|_x^\infty\right) dx\\ &= \int_0^\infty e^{-x} dx\\ &= 1.\end{aligned}$$

Here we used the answer of Problem 5.

Method 2:

$$\begin{aligned}\int_0^\infty \int_x^\infty e^{-y} dy dx &= \int_0^\infty \int_0^y e^{-y} dx dy\\ &= \int_0^\infty e^{-y} \int_0^y dx dy\\ &= \int_0^\infty y e^{-y} dy\\ &= 1.\end{aligned}$$

Here we used the answer of Problem 6. (Or we could solve Problem 6 from Problem 7 without intergration by parts.)

8. $\displaystyle\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}.$

Solution: There are two ways to see this. The classical approach is to observe that

$$\begin{aligned}\left(\int_{-\infty}^\infty e^{-x^2} dx\right)^2 &= \int_{-\infty}^\infty e^{-x^2} dx \int_{-\infty}^\infty e^{-y^2} dy\\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)} dx dy\\ &= \int_0^{2\pi} \int_0^\infty r e^{-r^2} dr d\theta\\ &= \int_0^{2\pi} \frac{1}{2} d\theta\\ &= \pi.\end{aligned}$$

Here we used change of cordinate from the Cartesian (x, y) to the polar $(r\cos\theta, r\sin\theta)$ with $dxdy = rdrd\theta$.

On the other hand, we know from ECE 109 that a Gaussian random variable with zero mean and variance σ^2 has the density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}},$$

which integrates to 1. Thus by taking $\sigma^2 = 1/2$, we have

$$\int \frac{1}{\sqrt{\pi}} e^{-x^2} dx = 1.$$

9. $\displaystyle\int_{-\infty}^{\infty} x e^{-x^2} dx = 0.$

   Solution: Since the integrand $f(x) = xe^{-x^2}$ is odd, i.e., $f(x) = -f(-x)$, the integral is zero. (Or consider the mean of the corresponding Gaussian distribution.)

10. $\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}^{-1} = \dfrac{1}{1-r^2} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix}.$

    Solution: This comes from the well-known 2×2 matrix inversion formula

    $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

    (Note: There are many ways to find the inverse of the general $n \times n$ matrices. One of common methods is applying Gaussian elimination process to the original matrix and the identity matrix. Try Google or Wikipedia with the "Gaussian elimination".)

11. $\lim_{x\to 0} x \ln x = 0.$

    Solution:

    $$\begin{aligned}
    \lim_{x\to 0} x \ln x &= \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} \\
    &\overset{(a)}{=} \lim_{x\to 0} \frac{(\ln x)'}{\left(\frac{1}{x}\right)'} \\
    &= \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\
    &= \lim_{x\to 0} -x \\
    &= 0.
    \end{aligned}$$

    Here we applied the L'hopital's rule in (a).