UCSD ECE 153 Handout #27
Prof. Young-Han Kim Wednesday, December 10, 2008

Final Exam
(Total: 200 points)

There are 4 problems. The first 3 problems have 4 parts each, while the last problem has 8 parts. Each part is uniformly worth 10 points.

Your answer should be as clear, readable (and short) as possible. In particular, if the answer involves a pmf or pdf, make sure to identify the values or intervals for which the pmf or pdf is nonzero.

1. *Order statistics.*
   Let X_1, X_2, X_3 be independent and uniformly drawn from the interval $[0, 1]$. Let Y_1 be the smallest of X_1, X_2, X_3, let Y_2 be the median (second smallest) of X_1, X_2, X_3, and let Y_3 be the largest of X_1, X_2, X_3. For example, if $X_1 = .3, X_2 = .1, X_3 = .7$, then $Y_1 = .1, Y_2 = .3, Y_3 = .7$. The random variables Y_1, Y_2, Y_3 are called the *order statistics* of X_1, X_2, X_3.
   (a) What is the probability $\mathrm{P}\{X_1 \le X_2 \le X_3\}$?
   (b) Find the pdf of Y_1.
   (c) Find the pdf of Y_3.
   (d) (Difficult.) Find the pdf of Y_2.
   (Hint: $Y_2 \le y$ if and only if at least two among X_1, X_2, X_3 are $\le y$.)

2. *Fair coins.*
   We are given two coins: Coin 1 with bias (=probability of heads) 1/2 and Coin 2 with random bias $P \sim \mathrm{Unif}[0, 1]$. We pick one at random and flip it three times independently. The value of the bias does not change during the sequence of tosses. Let X be the number of heads.
   (a) Find the conditional pmf of X given that Coin 1 is selected.
   (b) Find the conditional pmf of X given that Coin 2 is selected.
   (c) Find the optimal decision rule $D(x) \in \{1, 2\}$ for deciding which coin is flipped such that the probability of decision error is minimized.
   (d) Find the associated probability of error.

3. *Estimation.*
Let $X \sim N(0, P)$ and $Z \sim N(0, N)$ are independent. Let $Y = X + Z$.

   (a) Find the minimum mean squared error (MSE) estimator of X given Y.

   (b) What is the associated MSE?

   (c) Find the minimum MSE estimator of X^2 given Y.

   (d) Find the minimum MSE *linear* estimator of X^2 given Y.
   (Hint: You can use symmetry to find $E(X^3)$ rather easily.)

4. *Autoregressive process.*
Let $X_0 \sim N(0, 1)$ and $X_n = \alpha X_{n-1} + Z_n$ for $n \geq 1$, where $Z_1, Z_2, \ldots$ are i.i.d. $\sim N(0, 1 - \alpha^2)$ and independent of X_0. Assume $-1 < \alpha < 1$.

   (a) Find the mean and autocorrelation function of X_n.

   (b) Is X_n a wide-sense stationary process?

   (c) Is X_n a Gaussian process?

   (d) Is X_n an independent-increment process?

   (e) Given the observation $X_1, X_2, \ldots, X_k$, find the minimum MSE *linear* estimator of X_{k+1}. (Hint: You might first consider the best *nonlinear* estimator.)

   (f) What is the associated MSE?

   (g) Given the observation $X_1, X_2, \ldots, X_k$, find the minimum MSE *linear* estimator of X_{k+2}.

   (h) What is the associated MSE?