UCSD ECE 153 Handout #6
Prof. Young-Han Kim Thursday, October 9, 2008

Solutions to Homework Set #1
(Prepared by TA Halyun Jeong)

1. Read Sections 2.1 to 2.5 in the text. Try to work on all examples.

2. *World Series.* The World Series is a seven-game series that terminates as soon as either team wins four games. Suppose San Diego Padres (denoted as A) and New York Yankees (denoted as B) match up in the series. Possible outcomes include AAAA, ABABABA, and BBBAAAA. Assume that each game is independent and both teams are equally strong.

   (a) Describe the sample space of all possible outcomes.

   (b) What is the probability that Padres will win the series?

   (c) What is the probability that all seven games will be played?

   (d) Suppose Padres lost the first three games. What is the (conditional) probability that they will still win the series?

   Solution:

   (a) The sample space consists of two situations: Padres win or Yankees win. First we consider the outcomes that Padres win the game. There are five possible cases.

      i. Padres don't lose at all.
      AAAA
      There is only one outcome for this case.

$$P\{\text{SD 4 - NY 0}\} = \frac{1}{2^4} = \frac{1}{16}$$

      ii. Padres lose one game.
      BAAAA, ABAAA, AABAA, AAABA
      In this case, there are five games and Padres must win the last game (why), Among first 4 games we can have arbitrary combination of 3A1B, so there are $\binom{4}{1} = 4$ outcomes.

$$P\{\text{SD 4 - NY 1}\} = \frac{4}{2^5} = \frac{4}{32} = \frac{1}{8}$$

      iii. Padres lose two games.
      BBAAAA, ABBAAA, AABBAA, AAABBA, BABAAA

ABABAA, AABABA, BAABAA, ABAABA, BAAABA
There are $\binom{5}{2} = 10$ outcomes for this case.

$$\text{P}\{\text{SD 4 - NY 2}\} = \frac{10}{2^6} = \frac{10}{64} = \frac{5}{32}$$

iv. Padres lose three games.
BBBAAAA, ABBBAAA, AABBBAA, AAABBBA, BBABAAA
ABBABAA, AABBABA, BABBAAA, ABABBAA, AABABBA
BBAABAA, ABBAABA, BBAAABA, BAABBAA, ABAABBA
BAAABBA, ABABABA, BABABAA, BAABABA, BABAABA
There are $\binom{6}{3} = 20$ outcomes for this case.

$$\text{P}\{\text{SD 4 - NY 3}\} = \frac{20}{2^7} = \frac{20}{128} = \frac{5}{32}$$

Vice versa, if Yankees win, the cases would be the same. So we can write our sample space as:

$$\Omega = \left\{ \begin{array}{c} AAAA \\ BAAAA, ABAAA, AABAA, AAABA \\ BBAAAA, ABBAAA, AABBAA, AAABBA, BABAAA \\ ABABAA, AABABA, BAABAA, ABAABA, BAAABA \\ BBBAAAA, ABBBAAA, AABBBAA, AAABBBA, BBABAAA \\ ABBABAA, AABBABA, BABBAAA, ABABBAA, AABABBA \\ BBAABAA, ABBAABA, BBAAABA, BAABBAA, ABAABBA \\ BAAABBA, ABABABA, BABABAA, BAABABA, BABAABA \\ BBBB \\ ABBBB, BABBB, BBABB, BBBAB \\ AABBBB, BAABBB, BBAABB, BBBAAB, ABABBB \\ BABABB, BBABAB, ABBABB, BABBAB, ABBBAB \\ AAABBBB, BAAABBB, BBAAABB, BBBAAAB, AABABBB \\ BAABABB, BBAABAB, ABAABBB, BABAABB, BBABAAB \\ AABBABB, BAABBAB, AABBBAB, ABBAABB, BABBAAB \\ ABBBAAB, BABABAB, ABABABB, ABBABAB, ABABBAB \end{array} \right\}.$$

(b) Define the event C that Padres win the series. Then

$$\begin{aligned} \text{P}(C) &= \text{P}\{\text{Padres wins the game}\} \\ &= \text{P}\{\text{SD 4 - NY 0}\} + \text{P}\{\text{SD 4 - NY 1}\} + \text{P}\{\text{SD 4 - NY 2}\} + \text{P}\{\text{SD 4 - NY 3}\} \\ &= \frac{1}{16} + \frac{1}{8} + \frac{5}{32} + \frac{5}{32} \\ &= \frac{1}{2} \end{aligned}$$

(c) Define the event D that all seven games will be played. Then

$$\begin{aligned} \mathrm{P}(D) &= \mathrm{P}\{\text{all the seven games will be played}\} \\ &= \mathrm{P}\{\text{NY 4 - SD 3}\} + \mathrm{P}\{\text{SD 4 - NY 3}\} \\ &= \frac{5}{32} + \frac{5}{32} \\ &= \frac{5}{16} \end{aligned}$$

(d) In this part, we would like to compute the probability that Padres will win the series (the event C) given that they lost the first three games, which is denoted by F). Then,

$$\begin{aligned} \mathrm{P}(C|F) &= \frac{\mathrm{P}(C \cap F)}{\mathrm{P}(F)} \\ &= \mathrm{P}\{\text{Padres will win the series} \mid \text{Padres lost the first three games}\} \\ &= \frac{\mathrm{P}\{\text{Padres lost the first three games but they still win the series}\}}{\mathrm{P}\{\text{Padres lost the first three games}\}} \\ &= \frac{1/128}{1/8} \\ &= \frac{1}{16}. \end{aligned}$$

3. *Monty Hall.* Gold is placed behind one of three curtains. A contestant chooses one of the curtains, Monty Hall (the game host) opens one of the unselected empty curtains. The contestant has a choice either to switch his selection to the third curtain or not.

   (a) What is the sample space for this random experiment? (Hint: An outcome consists of the curtain with gold, the curtain chosen by the contestant, and the curtain chosen by Monty.)

   (b) Assume that placement of the gold behind the three curtains is random, the contestant choice of curtains is random and independent of the gold placement, and that Monty Hall's choice of an empty curtain is random among the alternatives. Specify the probability measure for this random experiment and use it to compute the probability of winning the gold if the contestant decides to switch.

Solution:

(a) The sample space consists of triplets of the form (Curtain with Gold behind it, Curtain chosen by the Player, Curtain that Monty opens). We denote the curtains by A, B, and C. So we can write our sample space as:

$$\Omega = \left\{ \begin{array}{c} (A, A, B), (A, A, C), (A, B, C), (A, C, B), (B, B, A), (B, B, C), \\ (B, A, C), (B, C, A), (C, C, A), (C, C, B), (C, B, A), (C, A, B) \end{array} \right\}.$$

(b) As discussed in class, for a discrete sample space the probability measure can be completely specified by probabilities of the single outcome events. For this problem we can specify the probabilities as follows:

$$\begin{aligned}
\mathrm{P}\{(A,A,B)\} &= \mathrm{P}\{\text{Monty opens } B \mid \text{Gold behind } A\text{, player's first choice is } A\}\times \\
&\quad \mathrm{P}\{\text{Gold behind } A\text{, player's first choice is } A\} \\
&= \frac{1}{2}\mathrm{P}\{\text{Gold is behind } A\} \times \mathrm{P}\{\text{player's first choice is } A\} \\
&= \frac{1}{2} \times \frac{1}{3} \times \frac{1}{3} \\
&= \frac{1}{18}.
\end{aligned}$$

Note that the last action of Monty Hall is not independent of where the gold is placed and what the player's initial choice was. That's why we used conditional probability in the above derivation. Using similar arguments we get: $\mathrm{P}\{(A,A,C)\} = \mathrm{P}\{(B,B,A)\} = \mathrm{P}\{(B,B,C)\} = \mathrm{P}\{(C,C,A)\} = \mathrm{P}\{(C,C,B)\} = \frac{1}{18}$. This covers all the cases where the gold placement and the initial choice of contestant coincide. For the other cases

$$\begin{aligned}
\mathrm{P}\{(A,B,C)\} &= \mathrm{P}\{\text{Monty opens C} \mid \text{Gold behind } A\text{, player's first choice is } B\}\times \\
&\quad \mathrm{P}\{\text{Gold is behind } A\text{, player's first choice is } B\} \\
&= 1 \times \mathrm{P}\{\text{Gold is behind } A\} \times \mathrm{P}\{\text{player's first choice is } B\} \\
&= 1 \times \frac{1}{3} \times \frac{1}{3} \\
&= \frac{1}{9}.
\end{aligned}$$

Using similar arguments, $\mathrm{P}\{(A,C,B)\} = \mathrm{P}\{(B,A,C)\} = \mathrm{P}\{(B,C,A)\} = \mathrm{P}\{(C,A,B)\} = \mathrm{P}\{(C,B,A)\} = \frac{1}{9}$ and we now have a complete description of the probability space for this random experiment.

To find the probability of winning if the player decides to switch, we need to find the subset W of the sample space corresponding to this event, which is

$$W = \{(A,B,C),(A,C,B),(B,A,C),(B,C,A),(C,A,B),(C,B,A)\}.$$

But the probability of the set W is the sum of the probabilities of its members, and, therefore, the probability of winning is

$$\begin{aligned}
\mathrm{P}(W) &= \frac{1}{9} \times 6 \\
&= \frac{2}{3}.
\end{aligned}$$

4. A number X is selected uniformly at random in the interval $[-1,1]$. Let the events $A = \{X > 0\}, B = \{|X + 0.5| < 1\}$, and $C = \{X < -0.75\}$.
   (a) Find the probabilities of $B, A \cap B$, and $A \cap C$.
   (b) Find the probabilities of $A \cup B, A \cup C$, and $A \cup B \cup C$.

Solution: The probability measure can be specified by first assigning probabilities to the intervals in $[-1, 1]$. Since we are selecting the outcomes at random, the probability of an interval $[a, b]$ for some $-1 \leq a, b \leq 1$ is $\frac{1}{2}(b - a)$, and is the same regardless of whether the interval is closed, open, or half open. By expressing any other set as countable unions, intersections, and complements of intervals and using the axioms of probability, we can find the probability of any other (Borel) set.

(a) The probabilities are:

$$\begin{aligned} \mathrm{P}(B) &= \mathrm{P}((-1, 0.5)) = \frac{3}{4}, \\ \mathrm{P}(A \cap B) &= \mathrm{P}((0, 0.5)) = \frac{1}{4}, \\ \mathrm{P}(A \cap C) &= \mathrm{P}(\phi) = 0. \end{aligned}$$

(b) The probabilities are

$$\begin{aligned} \mathrm{P}(A \cup B) &= \mathrm{P}((-1, 1]) = 1, \\ \mathrm{P}(A \cup C) &= \mathrm{P}((0, 1] \cup [-1, -0.75)]) = \frac{5}{8}, \\ \mathrm{P}(A \cup B \cup C) &= \mathrm{P}([-1, 1]) = 1. \end{aligned}$$

5. Let A, B be two events with $\mathrm{P}(A) \geq \frac{1}{2}$ and $\mathrm{P}(B) \geq \frac{2}{3}$. Show that $\mathrm{P}(A \cap B) \geq \frac{1}{6}$.

Solution: We have

$$\begin{aligned} \mathrm{P}(A \cap B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B) \\ &\geq \mathrm{P}(A) + \mathrm{P}(B) - 1 \\ &= \frac{1}{2} + \frac{2}{3} - 1 \\ &= \frac{1}{6}. \end{aligned}$$

6. Show that the events A and B are independent if $\mathrm{P}(A|B) = \mathrm{P}(A|B^c)$.

Solution: It is given that

$$\mathrm{P}(A|B) = \mathrm{P}(A|B^c).$$

Now

$$\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(A \cap B^c)}{\mathrm{P}(B^c)}.$$

Therefore

$$\begin{aligned} \mathrm{P}(A \cap B)\mathrm{P}(B^c) &= \mathrm{P}(A \cap B^c)\mathrm{P}(B) \\ &= (\mathrm{P}(A) - \mathrm{P}(A \cap B))\,\mathrm{P}(B), \end{aligned}$$

which implies that

$$P(A \cap B)(P(B^c) + P(B)) = P(A \cap B) = P(A)P(B).$$

7. *A pair of dice.* Let $(X, Y) \in \{1, \dots, 6\} \times \{1, \dots, 6\}$ denote the outcome of rolling a pair of independent fair dice. Define the events A, B, C as follows:

$$A = \{X \text{ is odd, i.e., } X \in \{1,3,5\}\}, \quad B = \{X \in \{1,2,3\}\}, \quad C = \{X + Y = 9\}.$$

   (a) Find $P(A), P(B)$, and $P(C)$.
   (b) Show that $P(A \cap B \cap C) = P(A)P(B)P(C)$.
   (c) Are A, B, C statistically independent? (Hint: Compare $P(A \cap B)$ and $P(A)P(B)$.)

Solution:

(a) The probabilities are:

$$\begin{aligned}
P(A) &= P(\{X \in \{1,3,5\}\}) = 3 \times \frac{1}{6} = \frac{1}{2}, \\
P(B) &= P(\{X \in \{1,2,3\}\}) = 3 \times \frac{1}{6} = \frac{1}{2}, \text{ and} \\
P(C) &= P(\{X + Y = 9\}) \\
&= P(\{(X, Y) \in \{(3,6),(4,5),(5,4),(6,3)\}\}) = 4 \times \frac{1}{36} = \frac{1}{9}.
\end{aligned}$$

(b) The probabilities are:

$$\begin{aligned}
P(A \cap B \cap C) &= P((A \cap B) \cap C) = P((\{X \in \{1,3,5\}\} \cap \{X \in \{1,2,3\}\}) \cap C) \\
&= P((\{X \in \{1,3\}\}) \cap \{(X, Y) \in \{(3,6),(4,5),(5,4),(6,3)\}\}) \\
&= P(\{(X, Y) \in \{(3,6)\}\} = \frac{1}{36}. \\
P(A)P(B)P(C) &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{9} = \frac{1}{36}.
\end{aligned}$$

Thus, $P(A \cap B \cap C) = P(A)P(B)P(C)$.

(c) No. A, B, C are not statistically independent since $P(A \cap B)$ and $P(A)P(B)$ are not equal. We can easily compute these probabilities as follows.

$$\begin{aligned}
P(A \cap B) &= P(\{X \in \{1,3,5\}\} \cap \{X \in \{1,2,3\}\}) \\
&= P(\{X \in \{1,3\}\}) = 2 \times \frac{1}{6} = \frac{1}{3} \\
P(A)P(B) &= \frac{1}{2} \times \frac{1}{2} \quad \text{(from part (a))} \\
&= \frac{1}{4}
\end{aligned}$$

8. *Negative evidence.* Suppose that the evidence of an event B increases the probability of a criminal's guilt; that is, if A is the event that the criminal is guilty, then $\mathrm{P}(A|B) \geq \mathrm{P}(A)$. Does the absence of the event B decrease the criminal's probability of being guilty? In other words, is $\mathrm{P}(A|B^c) \leq P(A)$? Prove or provide a counterexample.

   Solution: We know that $\mathrm{P}(A|B) \geq \mathrm{P}(A)$. From the law of total probability, we have

   $$\mathrm{P}(A) = \mathrm{P}(A|B)\mathrm{P}(B) + \mathrm{P}(A|B^c)\mathrm{P}(B^c).$$

   Thus

   $$\begin{aligned} \mathrm{P}(A|B^c)\mathrm{P}(B^c) &= \mathrm{P}(A) - \mathrm{P}(A|B)\mathrm{P}(B) \\ &\leq \mathrm{P}(A)(1 - \mathrm{P}(B)) \\ &= \mathrm{P}(A)\mathrm{P}(B^c). \end{aligned}$$

   Finally, dividing both sides by $\mathrm{P}(B^c)$, we can conclude that $\mathrm{P}(A|B^c) \leq \mathrm{P}(A)$; that is, the absence of a positive evidence is a negative evidence.

9. A ternary communication channel is shown in the Figure 1. Suppose that the input symbols 0,1, and 2 occur with probability $\frac{1}{2}$,$\frac{1}{3}$ and $\frac{1}{6}$ respectively.

   (a) Find the probabilities of the output symbols.

   (b) Suppose that a 1 is observed as an output. What is the probability that the input was 0? 1? 2?

   Your answers should be in terms of the conditional error probability ϵ.

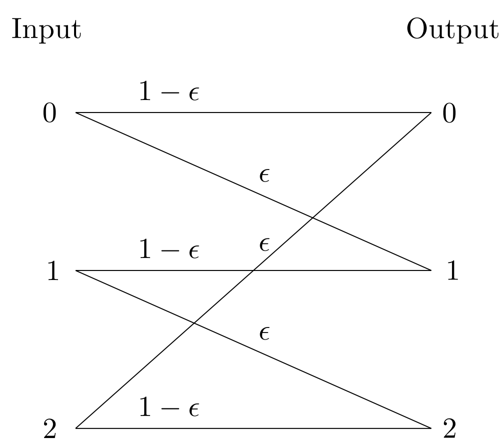

Figure 1: Ternary Communication Channel

Solution:

(a) Denote the output by Y and the input by X. Using the law of total probability and conditional probability we get

$$\begin{aligned}
\mathrm{P}\{Y=0\} &= \mathrm{P}\{Y=0|X=0\}\mathrm{P}\{X=0\}+\mathrm{P}\{Y=0|X=2\}\mathrm{P}\{X=2\} \\
&= \frac{1}{2}(1-\epsilon)+\frac{1}{6}\epsilon=\frac{1}{2}-\frac{1}{3}\epsilon, \\
\mathrm{P}\{Y=1\} &= \frac{1}{2}\epsilon+\frac{1}{3}(1-\epsilon)=\frac{1}{3}+\frac{1}{6}\epsilon, \\
\mathrm{P}\{Y=2\} &= \frac{1}{3}\epsilon+\frac{1}{6}(1-\epsilon)=\frac{1}{6}+\frac{1}{6}\epsilon.
\end{aligned}$$

Note that the sum of the probabilities is 1.

(b) The conditional probabilities are

$$\begin{aligned}
\mathrm{P}\{X=0|Y=1\} &= \frac{\mathrm{P}\{X=0,Y=1\}}{\mathrm{P}\{Y=1\}}=\frac{\frac{1}{2}\epsilon}{\frac{1}{3}+\frac{1}{6}\epsilon}=\frac{3\epsilon}{2+\epsilon}, \\
\mathrm{P}\{X=1|Y=1\} &= \frac{\frac{1}{3}(1-\epsilon)}{\frac{1}{3}+\frac{1}{6}\epsilon}=\frac{2-2\epsilon}{2+\epsilon}, \\
\mathrm{P}\{X=2|Y=1\} &= 0.
\end{aligned}$$

Again, note that the sum of the conditional probabilities is 1.