UCSD ECE 153 | Handout #8
Prof. Young-Han Kim | Thursday, October 16, 2008

**Solutions to Homework Set #2**
(Prepared by TA Halyun Jeong)

1. Read Sections 3.1–3.2, 3.4–3.6, 4.1–4.2, 4.4–4.5, 4.7, 4.9, 5.1-5.5, 5.7-5.8 in the text. Try to work on all examples.

2. *Juror's fallacy.* Suppose that $P(A|B) \geq P(A)$ and $P(A|C) \geq P(A)$. Is it always true that $P(A|B,C) \geq P(A)$ ? Prove or provide a counterexample.

   **Solution:** The answer is no. There are many counterexamples that can be given. For example, suppose a fair die is thrown and let $X$ denote the number of dots. Let $A$ be the event that $X = 3$ or 6; let $B$ be the event that $X = 3$ or 5; and let $C$ be the event that $X = 5$ or 6.

   Then, we have
   $$P(A) = 1/3, \qquad P(A|B) = P(A|C) = 1/2, \qquad \text{but } P(A|B,C) = 0.$$
   Apparently, having two positive evidences does not necessarily lead to a stronger evidence.

3. Let $X$ be a geometric random variable with pmf
   $$p_X(k) = p(1-p)^{k-1}, \;\; k = 1, 2, \ldots.$$
   Find and plot the conditional pmf $p_X(k|A) = P\{X = k | X \in A\}$ if:

   (a) $A = \{X > m\}$ where $m$ is a positive integer.
   (b) $A = \{X < m\}$.
   (c) $A = \{X \text{ is an even number}\}$.

   Comment on the shape of the conditional pmf of part (a).

   **Solution:**

   (a) We have
   $$\begin{aligned} P(A) &= \sum_{n=m+1}^{\infty} p(1-p)^{n-1} \\ &= \sum_{n=0}^{\infty} p(1-p)^{n+m} \\ &= p(1-p)^m \sum_{n=0}^{\infty} (1-p)^n \\ &= (1-p)^m. \end{aligned}$$

For $k \leq m$, $p_X(k|A) = 0$. For $k > m$,

$$\begin{aligned} p_X(k|A) &= P\{X = k|X > m\} \\ &= \frac{P\{X = k\}}{P\{X > m\}} \\ &= \frac{p(1-p)^{k-1}}{(1-p)^m} \\ &= p(1-p)^{k-m-1}. \end{aligned}$$

(b)

$$\begin{aligned} P(A) &= \sum_{n=0}^{m-2} p(1-p)^n \\ &= p\frac{1-(1-p)^{m-1}}{1-(1-p)} \\ &= 1-(1-p)^{m-1}. \end{aligned}$$

For $k \leq m$ or $k \leq 0$, $p_X(k|A) = 0$. For $k < m$

$$\begin{aligned} p_X(k|A) &= P\{X = k|X < m\} \\ &= \frac{P\{X = k\}}{P\{X < m\}} \\ &= \frac{p(1-p)^{k-1}}{1-(1-p)^{m-1}}. \end{aligned}$$

(c) We have

$$\begin{aligned} P(A) &= \sum_{n=0}^{\infty} p(1-p)((1-p)^2)^n \\ &= \frac{p(1-p)}{1-(1-p)^2} \\ &= \frac{1-p}{2-p}. \end{aligned}$$

Thus for $k$ even

$$\begin{aligned} p_X(k|A) &= P\{X = k|X \text{ is even}\} \\ &= \frac{P\{X = k\}}{P\{X \text{ is even}\}} \\ &= \frac{p(1-p)^{k-1}}{P(A)} \\ &= p(2-p)(1-p)^{k-2}. \end{aligned}$$

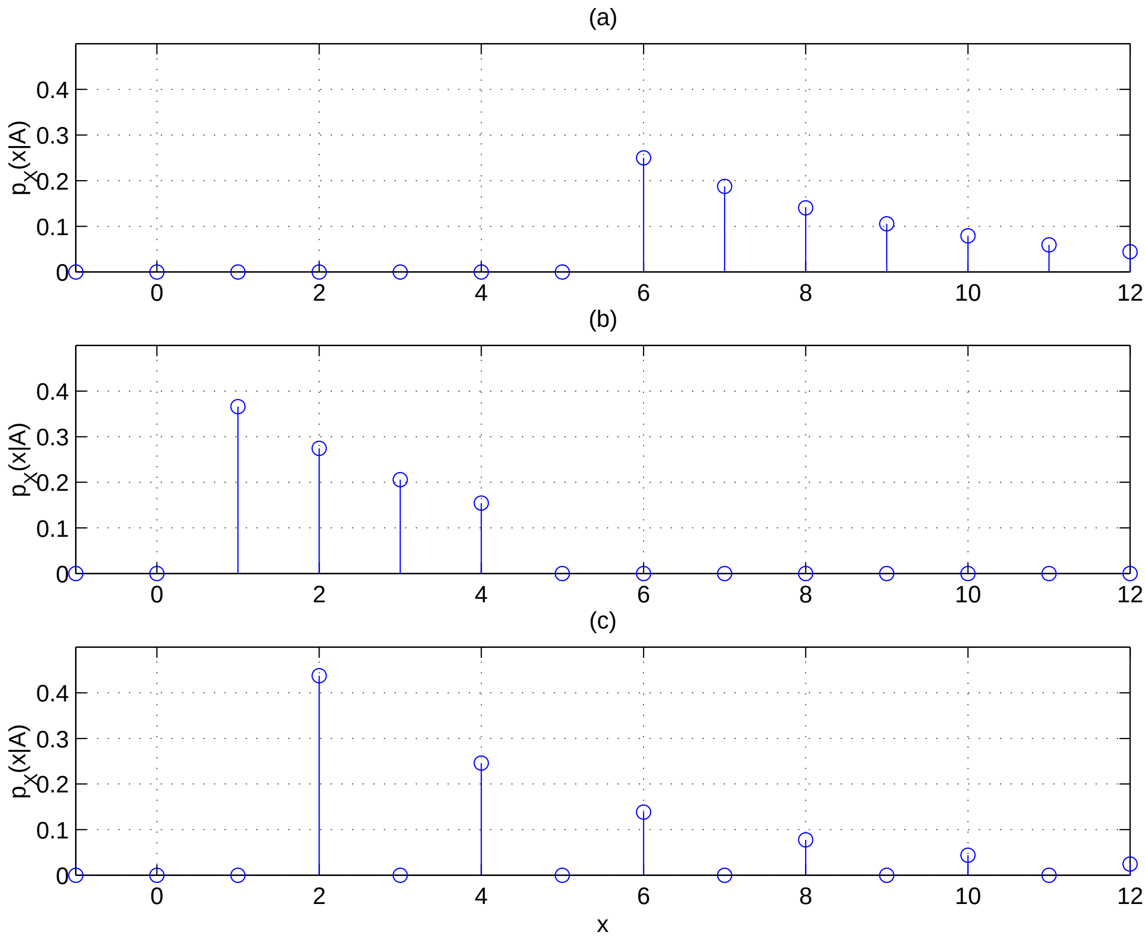


Figure 1: Plots of the conditional pmf's using $p = \frac{1}{4}$ and $m = 5$.

Plots are shown in Figure 1. The shape of the conditional pmf in part (a) shows that the geometric random variable is memoryless:

$$p_X(x|X > k) = p_X(x - k), \quad \text{for } x \geq k.$$

Note that in all three parts $p_X(x)$ is defined for *all* $x$. This is required.

4. *Negative binomial.* Suppose we observe an infinite sequence of independent coin flips with bias $p$ (i.e., the probability of heads is $p$ each time). Let $X$ be the number of coin flips until observing $k$ heads. Find the pmf of the random variable $X$.

   **Solution:** Consider the probability of observing a heads on the $x$-th trial preceded by $k-1$ heads and $x-k$ tails in some specific order. Since each trial is independent, we can multiply all the probabilities corresponding to each outcome. Each heads occurs with probability $p$ and each tail occurs with probability $1-p$. Therefore, the probability for the specified order, ending with heads is $p^{k-1}(1-p)^{x-k}p = p^k(1-p)^{x-k}$.
   The total number of possible outcomes in this coin toss composed of $k-1$ heads and $x-k$ tails,

with a heads on the last toss is $\begin{pmatrix} x-1 \\ k-1 \end{pmatrix}$. Since each sample point occurs mutually exclusive with equal probability $p^k(1-p)^{x-k}$, the pmf of the random variable $X$ is the multiplication of $\begin{pmatrix} x-1 \\ k-1 \end{pmatrix}$ and $p^k(1-p)^{x-k}$.

Therefore, the pmf of $X$ is $\begin{pmatrix} x-1 \\ k-1 \end{pmatrix} p^k(1-p)^{x-k}, \quad$ for $x = k, k+1, k+2 \ldots$.

5. Let $X \sim N(500, 400)$. Find

   (a) $\mathrm{P}\{480 < X < 520\}$.

   (b) $\mathrm{P}\{X < 540 | X > 460\}$.

   Hint: Use the $Q(\cdot)$ function table in the text (p. 169).

   **Solution:**

   (a) We can use the fact that $F_X(x) = \mathrm{P}\{X \le x\} = \Phi(\frac{x-m}{\sigma})$ and $Q(x) = 1 - \Phi(x)$ as in text. Here $m = 500$ and $\sigma = \sqrt{400} = 20$. Thus, $\mathrm{P}\{480 < X < 520\} = F_X(520) - F_X(480) = \Phi(\frac{520-500}{20}) - \Phi(\frac{480-500}{20}) = \Phi(1) - \Phi(-1) = Q(-1) - Q(1)$.
   By referring to the table 4.2 in the text, $Q(1) = 1.59 \times 10^{-1}$. Since $Q(-1) = 1 - Q(1) = 1 - 1.59 \times 10^{-1} \approx 0.84$, we have $\mathrm{P}\{480 < X < 520\} \approx 0.84 - 1.59 \times 10^{-1} \approx 0.68$.

   (b) $\mathrm{P}\{X < 540 | X > 460\} = \frac{\mathrm{P}\{460 < X < 540\}}{\mathrm{P}\{X > 460\}}$ from the definition of conditional probability.
   Using a similar argument as in part (a), we have $\mathrm{P}\{460 < X < 540\} = F_X(540) - F_X(460) = \Phi(\frac{540-500}{20}) - \Phi(\frac{460-500}{20}) = \Phi(2) - \Phi(-2) = Q(-2) - Q(2)$. Since $Q(-2) = 1 - Q(2) = 1 - 2.28 \times 10^{-2} \approx 0.98$, $\mathrm{P}\{460 < X < 540\} = Q(-2) - Q(2) \approx 0.98 - 2.28 \times 10^{-2} \approx 0.96$.
   Similarly, $\mathrm{P}\{X > 460\} = 1 - F_X(460) = 1 - \Phi(\frac{460-500}{20}) = 1 - \Phi(-2) = Q(-2) \approx 0.98$.
   Therefore, $\mathrm{P}\{X < 540 | X > 460\} = \frac{\mathrm{P}\{460 < X < 540\}}{\mathrm{P}\{X > 460\}} \approx \frac{0.96}{0.98} = 0.98$.

6. *Distance to the nearest star.* Let the random variable $N$ be the number of stars in a region of space of volume $V$. Assume that $N$ is a Poisson r.v. with pmf

$$p_N(n) = \frac{e^{-\rho V}(\rho V)^n}{n!}, \quad \text{for } n = 0, 1, 2, \ldots,$$

   where $\rho$ is the "density" of stars in space. We choose an arbitrary point in space and define the random variable $X$ to be the distance from the chosen point to the nearest star. Find the pdf of $X$ (in terms of $\rho$).

   **Solution:** The trick in this problem, as in many others, is to find a way to connect events regarding $X$ with events regarding $N$. In our case, for $x \ge 0$:

$$\begin{aligned} F_X(x) &= \mathrm{P}\{X \le x\} \\ &= 1 - \mathrm{P}\{X > x\} \\ &= 1 - \mathrm{P}\{\text{No stars within distance } x\} \\ &= 1 - \mathrm{P}\{N = 0 \text{ in sphere centered at origin of radius } x\} \\ &= 1 - e^{-\rho \frac{4}{3}\pi x^3}. \end{aligned}$$

Now differentiating, we get

$$f_X(x) = 4\pi\rho x^2 e^{-\rho\frac{4}{3}\pi x^3}.$$

For $x < 0$, both the cdf and the pdf are zero everywhere.

7. *Random phase signal.* Let $Y(t) = \sin(\omega t + \Theta)$ be a sinusoidal signal with random phase $\Theta \sim U[-\pi, \pi]$. Find the pdf of the random variable $Y(t)$ (assume here that both $t$ and the radial frequency $\omega$ are constant). Comment on the dependence of the pdf of $Y(t)$ on time $t$.

   **Solution:** We can easily see (by plotting $y$ vs. $\theta$) that for $y \in (-1, 1)$

$$\begin{aligned} P(Y \le y) &= P(\sin(\omega t + \Theta) \le y) \\ &= P(\sin(\Theta) \le y) \\ &= \frac{2\left(\sin^{-1}(y) + \frac{\pi}{2}\right)}{2\pi} \\ &= \sin^{-1}(y) + \frac{1}{2}. \end{aligned}$$

   By differentiating with respect to $y$, we get

$$f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}}.$$

   Note that $f_Y(y)$ does not depend on time $t$, i.e., is time invariant (or stationary) (more on this later in the course).

8. *Quantizer.* Let $X \sim \exp(\lambda)$, i.e., an exponential random variable with parameter $\lambda$ and $Y = \lfloor X \rfloor$, i.e., $Y = k$ for $k \le X < k+1$, $k = 0, 1, 2, \ldots$. Find the pmf of $Y$. Define the quantization error $Z = X - Y$. Find the pdf of $Z$.

   **Solution:** For $k < 0$, $p_Y(k) = 0$. Elsewhere

$$\begin{aligned} p_Y(k) &= P\{Y = k\} \\ &= P\{k \le X < k+1\} \\ &= F_X(k+1) - F_X(k) \\ &= \left(1 - e^{-\lambda(k+1)}\right) - \left(1 - e^{-\lambda k}\right) \\ &= e^{-\lambda k} - e^{-\lambda(k+1)} \\ &= e^{-\lambda k}\left(1 - e^{-\lambda}\right). \end{aligned}$$

   Check that $\sum_{k=0}^{\infty} e^{-\lambda k}\left(1 - e^{-\lambda}\right) = 1$.

   Since $Z = X - Y = X - \lfloor X \rfloor$ is the fractional part of $X$, $f_Z(z) = 0$ for $z < 0,\ z \ge 1$.

For $0 \le z < 1$, we have

$$\begin{aligned} F_Z(z) &= P(Z \le z) \\ &= \sum_{k=0}^{\infty} P(k \le X \le k+z) \\ &= \sum_{k=0}^{\infty} e^{-\lambda k} - e^{-\lambda(k+z)} \\ &= \frac{1-e^{-\lambda z}}{1-e^{-\lambda}}. \end{aligned}$$

By differentiating with respect to $z$, we get

$$f_Z(z) = \frac{\lambda e^{-\lambda z}}{1-e^{-\lambda}}$$

for $0 \le z \le 1$. Again, we can check $\int_0^1 \frac{\lambda e^{-\lambda z}}{1-e^{-\lambda}} dz = 1$.

9. *Gambling.* Alice enters a casino with one unit of capital. She looks at her watch to generate a uniform random variable $U \sim \text{unif}\,[0,1]$, then bets the amount $U$ on a fair coin flip. Her wealth is thus given by the r.v.

$$X = \begin{cases} 1+U, & \text{with probability } 1/2, \\ 1-U, & \text{with probability } 1/2. \end{cases}$$

Find the cdf of $X$.

**Solution:** For $x < 0$, the cdf is zero, and for $x > 2$ the cdf is one. In between, w e can find the cdf using the law of total probability. So consider

$$\begin{aligned} F_X(x) &= \mathrm{P}\{X \le x\} \\ &= \mathrm{P}\{X \le x, \mathrm{H}\} + \mathrm{P}\{X \le x, \mathrm{T}\} \\ &= \frac{1}{2}(\mathrm{P}\{X \le x | \mathrm{H}\} + \mathrm{P}\{X \le x | \mathrm{T}\}) \\ &= \frac{1}{2}(\mathrm{P}\{U \le x-1\} + \mathrm{P}\{U \ge 1-x\}) \\ &= \begin{cases} \frac{1}{2}(0+x), & 0 \le x \le 1 \\ \frac{1}{2}(x-\ 1+1), & 1 < x \le 2 \end{cases} \\ &= \frac{x}{2}, \text{ for } 0 < x \le 2. \end{aligned}$$

Thus $X \sim \text{Unif}[0,2]$.

10. Let the random variable $N(t)$ be the number of packets arriving during time $(0,t]$. Suppose $N(t)$ is Poisson with pmf

$$p_N(n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad \text{for } n = 0, 1, 2, \ldots.$$

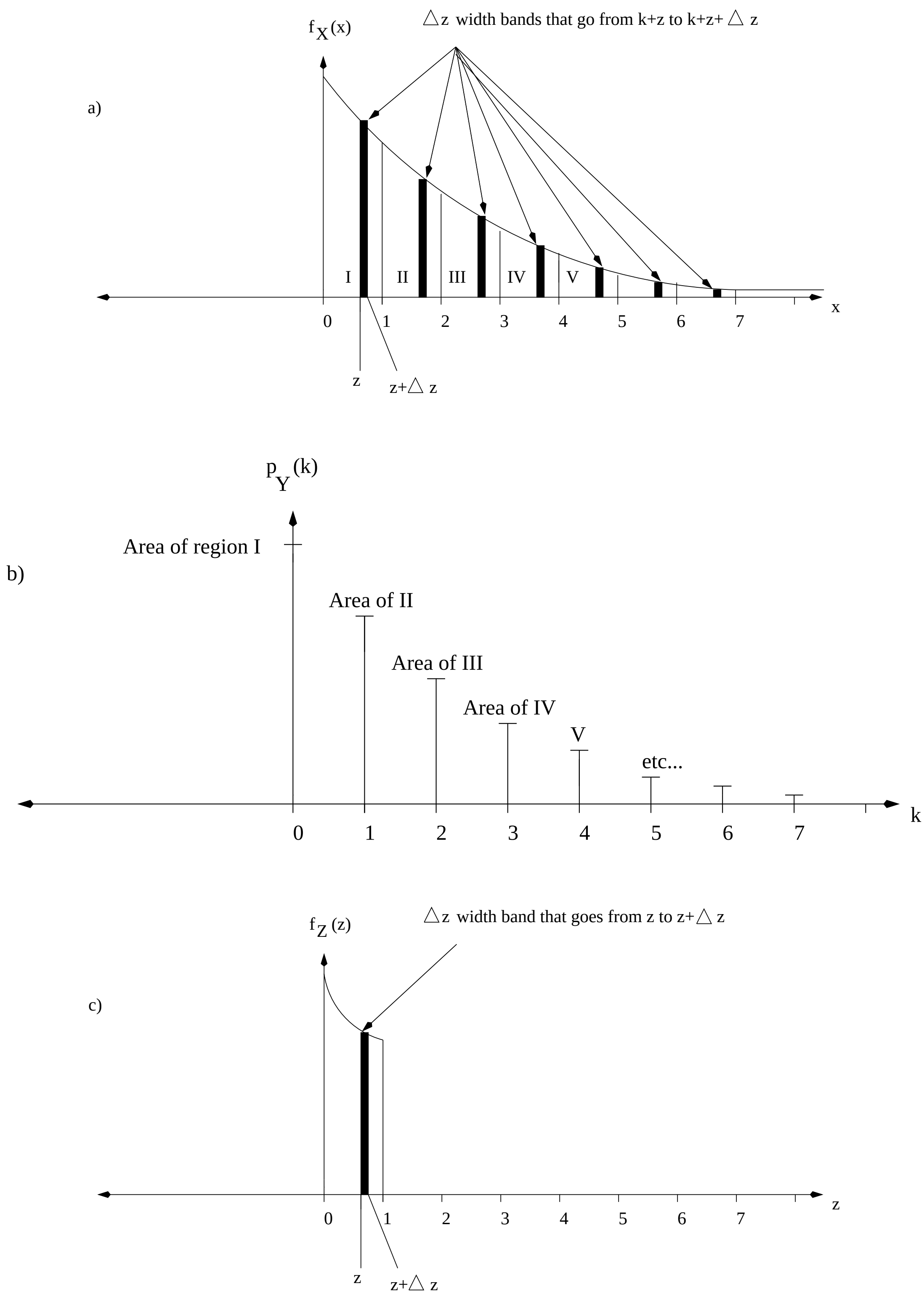


Figure 2: a) pdf of X, b) pmf of Y, c) pdf of Z

Let the random variable $X$ be the time to get the $n$-th packet. Find the pdf of $X$.

**Solution:** To find the density function $f_X(t)$ of the random variable $X$, note that the event $\{X \leq t\}$ occurs if the time of the $n$-th packet is in $[0, t]$, that is, if the number $N(t)$ of packets arriving in $[0, t]$ is at least $n$. Hence, the cdf $F_X(t)$ of $X$ is given by

$$F_X(t) = \mathrm{P}\{X \leq t\} = P\{N(t) \geq n\} = \sum_{k=n}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

Differentiating $F_X(t)$ with respect to $t$, we get the pdf $f_X(t)$ as

$$\begin{aligned} f_X(t) &= \sum_{k=n}^{\infty} \left[ -\lambda e^{-\lambda t} \frac{(\lambda t)^k}{k!} + \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!} \right] \\ &= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} - \sum_{k=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^k}{k!} + \sum_{k=n+1}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!} \\ &= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \end{aligned}$$

for $t > 0$. This is called the gamma distribution.

Or we can use another way. Since we know that the time interval $T$ between packet arrivals is an exponential random variable with pdf

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{if } t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $T_i$ denote the i.i.d. exponential interarrival times, then $X = T_1 + T_2 + ... + T_n$. By convolving $f_T(t)$ with itself $n - 1$ times, which can be also computed by its Fourier transform (characteristic function), we can show that the pdf of $X$ is given by

$$f_X(t) = \begin{cases} \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, & \text{if } t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

11. *First available teller.* Consider a bank with two tellers. The service times for the tellers are independent exponentially distributed random variables $X_1 \sim \mathrm{Exp}(\lambda_1)$ and $X_2 \sim \mathrm{Exp}(\lambda_2)$, respectively. You arrive at the bank and find that both tellers are busy but that nobody else is waiting to be served. You are served by the first available teller once he/she is free. What is the probability that you are served by the first teller?

**Solution:** From the memoryless property of the exponential distribution, the remaining services for the tellers are also independent exponentially distributed random variables with parameters $\lambda_1$ and $\lambda_2$, respectively. The probability that you will be served by the first teller is the probability that the first teller finishes the service before the second teller does. Thus,

$$
\begin{aligned}
\mathrm{P}\{X_1 < X_2\} &= \int_0^\infty \int_{x_1}^\infty \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} \, dx_2 \, dx_1 \\
&= \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2)x_1} \, dx_1 \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\end{aligned}
$$