Homework Set #3
Due: Thursday, October 24, 2008

1. Read Sections 6.5.1, 8.6.1–8.6.2 in the text. Try to work on all examples.

2. *Coin with random bias.* You are given a coin but are not told what its bias (probability of heads) is. You are told instead that the bias is the outcome of a random variable $P \sim \text{Unif}[0,1]$. To get more information about the coin bias, you flip it independently 10 times. Let X be the number of heads you get. Thus $X \sim \text{B}(10, P)$. Assuming that $X = 9$, find and sketch the *a posteriori* probability of P, i.e., $f_{P|X}(p|9)$.

3. *Signal or no signal (from Spring 2008 midterm).* Consider a communication system that is operated only from time to time. When the communication system is in the "normal" mode (denoted by $M = 1$), it transmits a random signal $S = X$ with

$$X = \begin{cases} +1, & \text{with probability } 1/2, \\ -1, & \text{with probability } 1/2. \end{cases}$$

When the system is in the "idle" mode (denoted by $M = 0$), it does not transmit any signal ($S = 0$). Both normal and idle modes occur with equal probability. Thus

$$S = \begin{cases} X, & \text{with probability } 1/2, \\ 0, & \text{with probability } 1/2. \end{cases}$$

The receiver observes $Y = S + Z$, where the ambient noise $Z \sim \text{Unif}[-1, 1]$ is independent of S.

(a) Find and sketch the conditional pdf $f_{Y|M}(y|1)$ of the receiver observation Y given that the system is in the normal mode.

(b) Find and sketch the conditional pdf $f_{Y|M}(y|0)$ of the receiver observation Y given that the system is in the idle mode.

(c) Find the optimal decoder $d(y)$ for deciding whether the system is normal or idle. Provide the answer in terms of intervals of y.

(d) Find the associated probability of error.

4. *Optical communication channel.* Let the signal input to an optical channel be given by

$$X = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ 10 & \text{with probability } \frac{1}{2}. \end{cases}$$

The conditional pmf of the output of the channel $Y|\{X=1\} \sim \text{Poisson}(1)$, i.e., Poisson with intensity $\lambda = 1$, and $Y|\{X=10\} \sim \text{Poisson}(10)$.

Show that the MAP rule reduces to

$$D(y) = \begin{cases} 1, & y < y^* \\ 10, & \text{otherwise.} \end{cases}$$

Find y^* and the corresponding probability of error.

5. *Maximal correlation.* Consider a pair of random variables (X,Y).

   (a) Show that $F_{X,Y}(x,y) \leq \min\{F_X(x), F_Y(y)\}$.

   Now let F and G be continuous and invertible cdf's and let $X \sim F$.

   (b) Find the distribution of $Y = G^{-1}(F(X))$.

   (c) Show that $F_{X,Y}(x,y) = \min\{F(x), G(y)\}$.

6. *Difference.* Let X and Y be continuous random variables with joint pdf

$$f(x,y) = \begin{cases} e^{-y}, & 0 \leq x \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Z = Y - X$. Find $f_{Z|X}(z|x)$.

7. *Max and min.* Let X and Y be two independent exponentially distributed random variables with the same parameter λ. Define $U = \max(X,Y)$, $V = \min(X,Y)$, and $W = U - V$.

   (a) Find the joint pdf of U and V.

   (b) Find the joint pdf of V and W. Are they independent ?

8. *First available teller.* Consider a bank with two tellers. The service times for the tellers are independent exponentially distributed random variables $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \text{Exp}(\lambda_2)$, respectively. You arrive at the bank and find that both tellers are busy but that nobody else is waiting to be served. You are served by the first available teller once he/she becomes free. Let the random variable Y denote your waiting time. Find the pdf of Y.