Solutions to Homework Set #3
(Prepared by TA Halyun Jeong)

1. Read Sections 6.5.1, 8.6.1–8.6.2 in the text. Try to work on all examples.

2. *Coin with random bias.* You are given a coin but are not told what its bias (probability of heads) is. You are told instead that the bias is the outcome of a random variable $P \sim \text{Unif}[0,1]$. To get more information about the coin bias, you flip it independently 10 times. Let X be the number of heads you get. Thus $X \sim \text{B}(10, P)$. Assuming that $X = 9$, find and sketch the *a posteriori* probability of P, i.e., $f_{P|X}(p|9)$.

   Solution: In order to find the conditional pdf of P, apply Bayes' rule for mixed random variables to get

$$f_{P|X}(p|x) = \frac{p_{X|P}(x|p)}{\int_0^1 p_{X|P}(x|p) f_P(p)\, dp} f_P(p).$$

   Now it is given that $X = 9$, thus for $0 \le p \le 1$

$$\begin{aligned} f_{P|X}(p|9) &= \frac{p^9(1-p)}{\int_0^1 p^9(1-p)\, dp} \\ &= \frac{p^9(1-p)}{\frac{1}{110}} \\ &= 110 p^9(1-p). \end{aligned}$$

   Figure 1 compares the unconditional and the conditional pdfs for P. It may be seen that given the information that 10 independent tosses resulted in 9 heads, the pdf is shifted towards the value $\frac{9}{10}$.

3. *Signal or no signal (from Spring 2008 midterm).* Consider a communication system that is operated only from time to time. When the communication system is in the "normal" mode (denoted by $M = 1$), it transmits a random signal $S = X$ with

$$X = \begin{cases} +1, & \text{with probability } 1/2, \\ -1, & \text{with probability } 1/2. \end{cases}$$

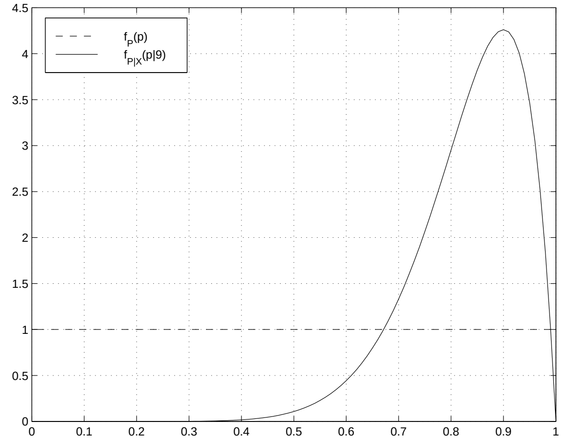


Figure 1: Comparison of *a priori* and *a posteriori* pdfs of P

When the system is in the "idle" mode (denoted by $M = 0$), it does not transmit any signal ($S = 0$). Both normal and idle modes occur with equal probability. Thus

$$S = \begin{cases} X, & \text{with probability } 1/2, \\ 0, & \text{with probability } 1/2. \end{cases}$$

The receiver observes $Y = S + Z$, where the ambient noise $Z \sim \text{Unif}[-1, 1]$ is independent of S.

(a) Find and sketch the conditional pdf $f_{Y|M}(y|1)$ of the receiver observation Y given that the system is in the normal mode.

(b) Find and sketch the conditional pdf $f_{Y|M}(y|0)$ of the receiver observation Y given that the system is in the idle mode.

(c) Find the optimal decoder $d(y)$ for deciding whether the system is normal or idle. Provide the answer in terms of intervals of y.

(d) Find the associated probability of error.

Solution:

(a) If $M = 1$,

$$Y = \begin{cases} 1 + Z, & \text{with probability } 1/2, \\ -1 + Z, & \text{with probability } 1/2. \end{cases}$$

Hence, we have

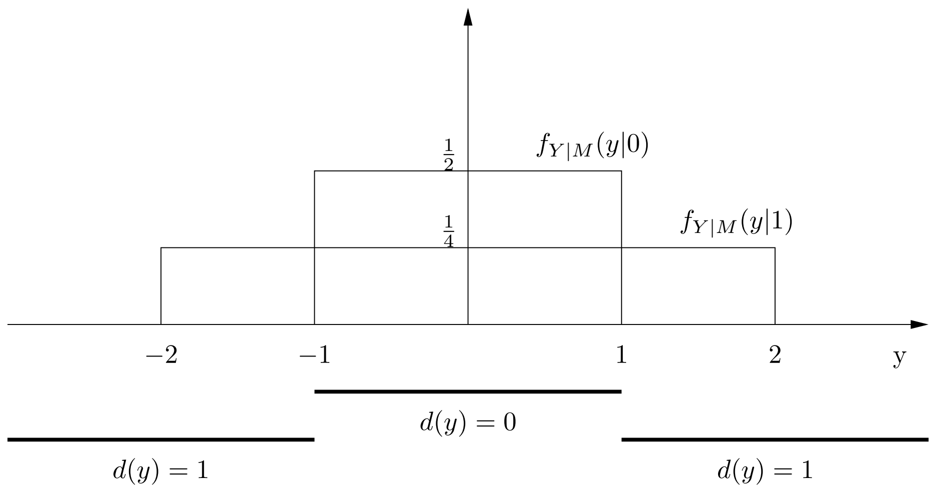
$$f_{Y|M}(y|1) = \begin{cases} \frac{1}{2} f_Z(y-1) + \frac{1}{2} f_Z(y+1) = \frac{1}{4}, & -2 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(b) If $M = 0$, $Y = Z$, so

$$f_{Y|M}(y|0) = \begin{cases} f_Z(y) = \frac{1}{2}, & -1 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Since both modes are equally likely, the optimal MAP decoding rule reduces to the ML rule, in which

$$d(y) = \begin{cases} 0, & \text{if } f_{Y|M}(y|0) > f_{Y|M}(y|1), \\ 1, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 0, & \text{if } -1 < y < 1, \\ 1, & \text{otherwise.} \end{cases}$$



(d) The probability of error is given by

$$\begin{aligned} \mathrm{P}\{M \ne d(Y)\} &= \mathrm{P}\{M = 1, -1 < Y < 1\} \\ &= \mathrm{P}\{M = 1\}\mathrm{P}\{-1 < Y < 1 | M = 1\} \\ &= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}. \end{aligned}$$

4. *Optical communication channel.* Let the signal input to an optical channel be given by

$$X = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ 10 & \text{with probability } \frac{1}{2}. \end{cases}$$

The conditional pmf of the output of the channel $Y|\{X = 1\} \sim \text{Poisson}(1)$, i.e., Poisson with intensity $\lambda = 1$, and $Y|\{X = 10\} \sim \text{Poisson}(10)$.

Show that the MAP rule reduces to

$$D(y) = \begin{cases} 1, & y < y^* \\ 10, & \text{otherwise.} \end{cases}$$

Find y^* and the corresponding probability of error.

Solution: The MAP rule

$$D(y) = \begin{cases} 1 & p_{X|Y}(1|y) > p_{X|Y}(10|y) \\ 10 & \text{otherwise} \end{cases}$$

minimizes the probability of decoding error. Since the *a priori* probabilities for the two X values are equal, the optimal decision rule is equivalent to the ML rule:

$$D(y) = \begin{cases} 1 & \frac{p_{Y|X}(y|1)}{p_{Y|X}(y|10)} > 1 \\ 10 & \text{otherwise.} \end{cases}$$

Here,

$$\begin{aligned} \frac{p_{Y|X}(y|1)}{p_{Y|X}(y|10)} &= \frac{e^{-1}/y!}{e^{-10}10^y/y!} \\ &= e^{9-y\ln(10)}. \end{aligned}$$

This ratio is greater than 1 if $y < \frac{9}{\ln(10)}$. Therefore,

$$D(y) = \begin{cases} 1 & y < \frac{9}{\ln(10)} \\ 10 & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} y^* &= \frac{9}{\ln(10)} \\ &= 3.91. \end{aligned}$$

The probability of error is

$$\begin{aligned} P_e &= \mathrm{P}\{D(Y) \neq X\} \\ &= \mathrm{P}\{Y > y^*|X = 1\}\mathrm{P}\{X = 1\} + \mathrm{P}\{Y < y^*|X = 10\}\mathrm{P}\{X = 10\} \\ &= \frac{1}{2}\sum_{y=4}^{\infty}\frac{e^{-1}}{y!} + \frac{1}{2}\sum_{y=0}^{3}\frac{e^{-10}10^y}{y!} \\ &= 0.0147. \end{aligned}$$

5. *Maximal correlation.* Consider a pair of random variables (X,Y).

   (a) Show that $F_{X,Y}(x, y) \leq \min\{F_X(x), F_Y(y)\}$.

Now let F and G be continuous and invertible cdf's and let $X \sim F$.

(b) Find the distribution of $Y = G^{-1}(F(X))$.

(c) Show that $F_{X,Y}(x, y) = \min\{F(x), G(y)\}$.

Solution:

(a) We have

$$F_{X,Y}(x, y) = P(X \le x, Y \le y) \le P(X \le x) = F_X(x),$$

and similarly, $F_{X,Y} \le F_Y(y)$. Thus,

$$F_{X,Y} \le \min\{F_X(x), F_Y(y)\}.$$

(b) We have

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(G^{-1}(F(X)) \le y) \\ &= P(F(X) \le G(y)) \\ &= P(X \le F^{-1}(G(y))) \\ &= F(F^{-1}(G(y))) \\ &= G(y). \end{aligned}$$

(c) We have

$$\begin{aligned} F_{X,Y}(x, y) &= P(X \le x, Y \le y) \\ &= P(X \le x, X \le F^{-1}(G(y))) \\ &= P(X \le \min\{x, F^{-1}(G(y))) \\ &= \min\{F(x), F(F^{-1}(G(y)))\} \\ &= \min\{F(x), G(y)\}. \end{aligned}$$

From part (a), this is the maximal joint cdf for any (X, Y) with the given marginal cdf's $F(x)$ and $G(y)$.

6. *Difference.* Let X and Y be continuous random variables with joint pdf

$$f(x, y) = \begin{cases} e^{-y}, & 0 \le x \le y, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Z = Y - X$. Find $f_{Z|X}(z|x)$.

Solution: First note that

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x, y)}{f_X(x)} \\ &= \frac{e^{-y}}{e^{-x}} \end{aligned} \qquad (1)$$

for $y \geq x \geq 0$.

Now for $x, z \geq 0$, we have

$$\begin{aligned} F_{Z|X}(z|x) &= \mathrm{P}\{Z \leq z|X = x\} \\ &= \mathrm{P}\{Y - X \leq z|X = x\} \\ &= \mathrm{P}\{Y - x \leq z|X = x\} \\ &= \mathrm{P}\{Y \leq z + x|X = x\} \\ &= F_{Y|X}(z + x|x). \end{aligned}$$

By differentiating with respect to z, we have from (1)

$$\begin{aligned} f_{Z|X}(z|x) &= f_{Y|X}(z + x|x) \\ &= \frac{e^{-(x+z)}}{e^{-x}} \\ &= e^{-z} \end{aligned}$$

for $z \geq 0$. Note that $f_{Z|X}(z|x) = 0$ for $z < 0$.

7. *Max and min.* Let X and Y be two independent exponentially distributed random variables with the same parameter λ. Define $U = \max(X, Y)$, $V = \min(X, Y)$, and $W = U - V$.

   (a) Find the joint pdf of U and V.

   (b) Find the joint pdf of V and W. Are they independent ?

Solution:

(a) We first find the joint cdf of U and V and then differentiate it to get the joinf pdf. But first note that both U and V are nonnegative random variables. Therefore, their joint pdf will be nonzero only in the first quadrant. Therefore, their joint pdf will be nonzero only in the first quadrant. We thus find the cdf for $u, v > 0$:

$$\begin{aligned} F_{U,V}(u, v) &= \mathrm{P}\{U \leq u, V \leq v\} \\ &= \mathrm{P}\{U \leq u\} - \mathrm{P}\{U \leq u, V > v\} \\ &= \mathrm{P}\{\max(X, Y) \leq u\} - \mathrm{P}\{\max(X, Y) \leq u, \min(X, Y) > v\} \\ &= \mathrm{P}\{X \leq u\}\mathrm{P}\{Y \leq u\} - \mathrm{P}\{v < X \leq u\}\mathrm{P}\{v < Y \leq u\}. \end{aligned}$$

When $u < v$, the second term is zero since the minimum cannot be greater than the maximum. So for the region $u < v$, $f_{U,V}(u, v) = 0$ since the derivative with respect to v is zero.

For $u \geq v$, we continue the above calculation to get

$$F_{U,V}(u, v) = (1 - e^{-\lambda u})^2 - (e^{-\lambda v} - e^{-\lambda u})^2.$$

Now taking partial derivatives with respect to u and v of $F_{U,V}(u,v)$ we obtain $f_{U,V}(u,v)$ for $u \geq v \geq 0$ as

$$f_{U,V}(u,v) = 2\lambda^2 e^{-\lambda(u+v)}.$$

(b) We present two ways of solving this problem.

Method 1: To find the joint pdf, we can first calculate the joint cdf of V and W as follows.

$$\begin{aligned} F_{W,V}(w,v) &= \mathrm{P}\{W \leq w, V \leq v\} \\ &= \mathrm{P}\{\max(X,Y) - \min(X,Y) \leq w, \min(X,Y) \leq v\} \\ &= \mathrm{P}\{Y - X \leq w, X \leq v | Y > X\}\mathrm{P}\{Y > X\} \\ &\quad + \mathrm{P}\{X - Y \leq w, Y \leq v | X \geq Y\}\mathrm{P}\{X \geq Y\}. \end{aligned}$$

Since X and Y are independent and identically distributed, it is easy to see (by symmetry!) that

$$\mathrm{P}\{Y \leq X\} = \mathrm{P}\{X < Y\} = \frac{1}{2}$$

and

$$\mathrm{P}\{Y - X \leq w, X \leq v | Y > X\} = \mathrm{P}\{X - Y \leq w, Y \leq v | Y \leq X\}.$$

Therefore,

$$F_{W,V}(w,v) = \mathrm{P}\{X - Y \leq w, Y \leq v | Y \leq X\}.$$

For $w, v \geq 0$, we have

$$\begin{aligned} F_{W,V}(w,v) &= \frac{\mathrm{P}\{X - Y \leq w, Y \leq v, Y \leq X\}}{\mathrm{P}\{Y \leq X\}} \\ &= \frac{\int_0^v \int_y^{y+w} f_{X,Y}(x,y)\, dx\, dy}{1/2} \\ &= 2\lambda^2 \int_0^v e^{-\lambda y} \int_y^{y+v} e^{-\lambda x}\, dx\, dy \\ &= 2\lambda \int_0^v e^{-2\lambda y}(1 - e^{-\lambda w})\, dy \\ &= (1 - e^{-2\lambda v})(1 - e^{-\lambda w}). \end{aligned}$$

Therefore for $w, v \geq 0$, by taking partial derivatives with respect to w and v we obtain

$$f_{W,V}(w,v) = 2\lambda^2 e^{-2\lambda v} e^{-\lambda w}.$$

For other regions, the joint pdf is zero since both V and W are positive random variables.

Finally, it is easy to check that $f_V(v) = 2\lambda e^{-2\lambda v}$ and $f_W(w) = \lambda e^{-\lambda w}$. Therefore, $f_{W,V}(w, v)$ is a product of the marginal pdfs and therefore W and V are independent.

Method 2: We can also find the joint pdf directly. Using the same argument as in Question 6, we have (for $u, w \geq 0$)

$$\begin{aligned} f_{W,V}(W, V) &= f_V(v) f_{W|V}(w|v) \\ &= f_V(v) f_{U|V}(w + v|v) \\ &= f_{U,V}(w + v, v). \end{aligned}$$

Now using the result of part (a), we get

$$f_{W,V}(w, v) = 2\lambda^2 e^{-2\lambda v} e^{-\lambda w} = f_W(w) f_V(v).$$

8. *First available teller.* Consider a bank with two tellers. The service times for the tellers are independent exponentially distributed random variables $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \text{Exp}(\lambda_2)$, respectively. You arrive at the bank and find that both tellers are busy but that nobody else is waiting to be served. You are served by the first available teller once he/she becomes free. Let the random variable Y denote your waiting time. Find the pdf of Y.

   Solution: First observe that $Y = \min(X_1, X_2)$. Since

$$\begin{aligned} \text{P}\{Y > y\} &= \text{P}\{X_1 > y, X_2 > y\} \\ &= \text{P}\{X_1 > y\}\text{P}\{X_2 > y\} \\ &= e^{-\lambda_1 y} \times e^{-\lambda_2 y} \\ &= e^{-(\lambda_1 + \lambda_2) y} \end{aligned}$$

   for $y \geq 0$, Y is an exponential random variable with pdf

$$f_Y(y) = \begin{cases} (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2) y}, & y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$