UCSD ECE 153 Handout #11
Prof. Young-Han Kim Thursday, October 23, 2008

Homework Set #4
Due: Thursday, October 30, 2008

1. Read Sections 4.3, 4.6, 5.7, 5.9, 6.5 in the text. Try to work on all examples.

2. *Two envelopes.* An amount A is placed in one envelope and the amount $2A$ is placed in another envelope. The amount A is fixed but unknown to you. The envelopes are shuffled and you are given one of the envelopes at random. Let X denote the amount you observe in this envelope. Designate by Y the amount in the other envelope. Thus

$$(X, Y) = \begin{cases} (A, 2A), & \text{with probability } \frac{1}{2}, \\ (2A, A), & \text{with probability } \frac{1}{2}. \end{cases}$$

You may keep the envelope you are given, or you can switch envelopes and receive the amount in the other envelope.

(a) Find $E(X)$ and $E(Y)$.
(b) Find $E\left(\frac{X}{Y}\right)$.
(c) Suppose you switch. What is the expected amount you receive?

3. *Tall trees.* Suppose that the average height of trees on campus is 20 feet. Argue that no more than half of the tree population is taller than 40 feet.

4. Let Λ and X be two random variables with

$$\Lambda \sim f_\Lambda(\lambda) = \begin{cases} \frac{5}{3}\lambda^{\frac{2}{3}}, & 0 \le \lambda \le 1 \\ 0, & \text{otherwise,} \end{cases}$$

and $X|\{\Lambda = \lambda\} \sim \text{Exp}(\lambda)$. Find $E(X)$.

5. *Inequalities.* Label each of the following statements with $=$, $\le$, or $\ge$. Justify each answer.

(a) $\frac{1}{E(X^2)}$ vs. $E\left(\frac{1}{X^2}\right)$.
(b) $(E(X))^2$ vs. $E(X^2)$.
(c) $\text{Var}(X)$ vs. $\text{Var}(E(X|Y))$.
(d) $E(X^2)$ vs. $E((E(X|Y))^2)$.

6. Let X and Y be two random variables with joint pdf

$$f(x,y) = \begin{cases} x+y, & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the MMSE estimator of X given Y.

   (b) Find the corresponding MSE.

   (c) Find the pdf of $Z = E(X|Y)$.

7. *Orthogonality.* Let $\hat{X}$ be the minimum MSE estimate of X given Y.

   (a) Show that for any function $g(y)$, $E((X - \hat{X})g(Y)) = 0$, i.e., the error $(X - \hat{X})$ and $g(Y)$ are orthogonal.

   (b) Show that

   $$\mathrm{Var}(X) = E(\mathrm{Var}(X|Y)) + \mathrm{Var}(\hat{X}).$$

   Provide a geometric interpretation for this result.

8. *Additive shot noise channel.* Consider an additive noise channel $Y = X + Z$, where the signal $X \sim N(0,1)$, and the noise $Z|\{X = x\} \sim N(0, |x|)$, i.e., the variance of the noise increases linearly with the absolute value of the signal.

   (a) Find $E(Z^2)$. We are expecting a numerical answer here.

   (b) Find the best linear MSE estimate of X given Y.

9. *Jointly Gaussian random variables.* Let X and Y be jointly Gaussian random variables with pdf

$$f_{X,Y}(x,y) = \frac{1}{\pi\sqrt{1/12}} e^{-2x^2-8y^2-4xy+12x+24y-24}.$$

   (a) Find $E(X)$, $E(Y)$, $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$, and $\mathrm{Cov}(X,Y)$.

   (b) Find the minimum MSE estimate of X given Y and its MSE.

10. Consider a channel with the observation $Y = XZ$, where the signal X and the noise Z are uncorrelated Gaussian random variables. Let $E(X) = 1$, $E(Z) = 3$, $\sigma_X^2 = 3$, and $\sigma_Z^2 = 8$.

   (a) Find the best MSE linear estimate of X given Y.

   (b) Suppose your friend from Caltech tells you that he was able to derive an estimator with a lower MSE. Your friend from UCLA disagrees, saying that this is not possible because the signal and the noise are Gaussian, and hence the linear MSE estimator will also be the best MSE estimator. Could your UCLA friend be wrong?