Homework Set #5
Due: Thursday, November 20, 2008

1. Work on the midterm problems to make sure you understand everything clearly.
2. Read Sections 6.1–6.5 in the text. Try to work on all examples.
3. Which of the following matrices can be a covariance matrix? Justify your answer either by constructing a random vector $\mathbf{X}$, as a function of the i.i.d. zero mean unit variance random variables Z_1, Z_2, and Z_3, with the given covariance matrix, or by establishing a contradiction.

$$\text{(a)} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}$$

4. Given a Gaussian random vector $\mathbf{X} \sim N(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} = (1\,2\,3)^T$ and

$$\Sigma = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

   (a) What is $P(X_1 + X_2 + 2X_3 < 0)$?

   (b) Find the joint pdf on $\mathbf{Y} = A\mathbf{X}$, where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}.$$

5. *Packet switching.* Let $N \sim \mathrm{P}(\lambda)$, i.e., Poisson with parameter λ, be the number of packets arriving at a switch per unit time. Each packet is routed to Output Port 1 with probability p and to Output Port 2 with probability $(1-p)$ independent of N and of other packets. Let X be the number of packets routed to Output Port 1 per unit time. Thus $X = 0$ if $N = 0$ and $X = \sum_{i=1}^{N} Z_i$ for $N > 0$, where

$$Z_i = \begin{cases} 1, & \text{packet } i \text{ routed to Port 1} \\ 0, & \text{packet } i \text{ routed to Port 2,} \end{cases}$$

   and $Z_1, Z_2, \ldots, Z_N$ are conditionally independent given N.

   (a) Find the mean and variance of X.

   (b) Find the pmf of X. What is the pmf of $N - X$?

6. *Estimation (from Spring 2008 final).* Let X_1 and X_2 be independent identically distributed random variables. Let $Y = X_1 + X_2$.

   (a) Find $E[X_1 - X_2|Y]$.

   (b) Find the minimum mean squared error estimate of X_1 given an observed value of $Y = X_1 + X_2$. (Hint: Consider $E[X_1 + X_2|X_1 + X_2]$.)

7. *Gaussian Markov chain (from Spring 2007 final).* Let X, Y, and Z be jointly Gaussian random variables with zero mean and unit variance, i.e., $EX = EY = EZ = 0$ and $EX^2 = EY^2 = EZ^2 = 1$. Let $\rho_{X,Y}$ denote the correlation coefficient between X and Y, and let $\rho_{Y,Z}$ denote the correlation coefficient between Y and Z. Suppose that X and Z are conditionally independent given Y.

   (a) Find $\rho_{X,Z}$ in terms of $\rho_{X,Y}$ and $\rho_{Y,Z}$.

   (b) Find the MMSE estimate of Z given (X, Y) and the corresponding MSE.

8. *Prediction of an autoregressive process.* Let $\mathbf{X} = [X_1 X_2 \ldots X_n]^T$ be a random vector with zero mean and covariance matrix

$$\Sigma_{\mathbf{X}} = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ \alpha & 1 & \alpha & & \\ \alpha^2 & \alpha & 1 & & \\ \vdots & & & \ddots & \\ \alpha^{n-1} & & & \cdots & 1 \end{bmatrix}$$

   for $|\alpha| < 1$. Given the observation $X_1, X_2, \ldots, X_{n-1}$, find the best linear MSE estimate (predictor) of X_n. Compute its MSE.

9. *Noise cancellation.* A classical problem in statistical signal processing involves estimating a weak signal (e.g., the heart beat of a fetus) in the presence of a strong interference (the heart beat of its mother) by making two observations; one with the weak signal present and one without (by placing one microphone on the mother's belly and another close to her heart). The observations can then be combined to estimate the weak signal by "cancelling out" the interference. The following is a simple version of this application.

   Let the weak signal X be a random variable with mean μ and variance P, and the observations be $Y_1 = X + Z_1$ (Z_1 being the strong interference), and $Y_2 = Z_1 + Z_2$ (Z_2 is a measurement noise), where Z_1 and Z_2 are zero mean with variances N_1 and N_2, respectively. Assume that X, Z_1 and Z_2 are uncorrelated. Find the best linear MSE estimate of X given Y_1 and Y_2 and its MSE. Interprete the results.