UCSD ECE 153 Handout #24
Prof. Young-Han Kim Thursday, December 4, 2008

Solutions to Homework Set #6
(Prepared by TA Halyun Jeong)

1. Read Sections 7.1–7.4, 9.1–9.6, 10.1–10.2, 10.4 in the text. Try to work on all examples.

2. *Symmetric random walk.* Let X_n be a random walk defined as

$$X_0 = 0$$
$$X_n = \sum_{i=1}^{n} Z_i,$$

where $Z_1, Z_2, \ldots$ are i.i.d. with $\mathrm{P}(Z_1 = -1) = \mathrm{P}(Z_1 = 1) = \frac{1}{2}$.

(a) Find $\mathrm{P}\{X_{10} = 10\}$.

(b) Find $\mathrm{P}\{\max_{1 \le i < 20} X_i = 10 | X_{20} = 0\}$.

(c) Find $\mathrm{P}\{X_n = k\}$.

Solution:

(a) Since the event $\{X_{10} = 10\}$ is equivalent to $\{Z_1 = \cdots = Z_{10} = 1\}$, we have $\mathrm{P}\{X_{10} = 10\} = 2^{-10}$.

(b) Since the event $\{\max X_i = 10, X_{20} = 0\}$ is equivalent to $\{Z_1 = \cdots = Z_{10} = 1, Z_{11} = \cdots = Z_{20} = -1\}$, we have

$$\mathrm{P}\{\max_{1 \le i < 20} X_i = 10 | X_{20} = 0\} = \frac{2^{-20}}{P(X_{20} = 0)} = \frac{1}{\binom{20}{10}}.$$

(c) As shown in class,

$$\mathrm{P}\{X_n = k\} = \mathrm{P}\{(n+k)/2 \text{ heads in } n \text{ independent coin tosses}\}$$
$$= \binom{n}{\frac{n+k}{2}} 2^{-n}$$

for $-n \le k \le n$ with $n + k$ even.

3. *Moving average process.* Let $Y_n = \frac{1}{2}Z_{n-1} + Z_n$ for $n \geq 1$, where $Z_0, Z_1, Z_2, \ldots$ are i.i.d. $\sim \mathcal{N}(0,1)$. Find the mean and autocorrelation function of Y_n.

   Solution:

$$EY_n = \frac{1}{2}EZ_{n-1} + EZ_n = 0.$$

$$\begin{aligned}
R_Y(m,n) &= E(Y_m Y_n) \\
&= E\left[\left(\frac{1}{2}Z_{n-1} + Z_n\right)\left(\frac{1}{2}Z_{m-1} + Z_m\right)\right] \\
&= \begin{cases} \frac{1}{2}E[Z_{n-1}^2], & n-m=1 \\ \frac{1}{4}E[Z_{n-1}^2] + E[Z_n^2], & n=m \\ \frac{1}{2}E[Z_n^2], & m-n=1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{5}{4}, & n=m \\ \frac{1}{2}, & |n-m|=1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

4. *Gauss-Markov process.* Let $X_0 = 0$ and $X_n = \frac{1}{2}X_{n-1} + Z_n$ for $n \geq 1$, where $Z_1, Z_2, \ldots$ are i.i.d. $\sim \mathcal{N}(0,1)$. Find the mean and autocorrelation function of X_n.

   Solution: It is easy to show that $E(X_n) = E(X_{n-1}) = \cdots = E(X_0) = 0$. (Why?) For the autocorrelation function, first note that

$$\begin{aligned}
E(X_n^2) &= \frac{1}{4}E(X_{n-1}^2) + E(Z_n^2) + E(X_{n-1}Z_n) \\
&= \frac{1}{4}E(X_{n-1}^2) + 1.
\end{aligned}$$

   Therefore, $EX_1^2 = \frac{1}{4} \cdot 0 + 1 = 1$. Similarly, $EX_2^2 = \frac{1}{4}E(X_1^2) + 1 = \frac{1}{4} \cdot 1 + 1 = \frac{5}{4}$. In general, $EX_n^2 = \sum_{i=0}^{n-1} \frac{1}{4^i} = \frac{4}{3}(1 - (\frac{1}{4})^{n+1})$. Moreover, we can write

$$X_n = \frac{1}{2^k}X_{n-k} + \frac{1}{2^{k-1}}Z_{n-k+1} + \frac{1}{2^{k-2}}Z_{n-k+2} + \cdots + Z_n,$$

   and thus

$$R_X(n, n-k) = EX_nX_{n-k} = \frac{1}{2^k}EX_{n-k}^2 = \frac{4}{3}\left(1 - \left(\frac{1}{4}\right)^{n-k+1}\right) \cdot \frac{1}{2^k}$$

   for all n and all $k \geq 0$.

5. *Discrete-time Wiener process.* Let Z_n, $n \geq 0$ be a discrete time white Gaussian noise (WGN) process, i.e., $Z_1, Z_2, \ldots$ are i.i.d. $\sim \mathcal{N}(0,1)$. Define the process X_n, $n \geq 1$ as $X_0 = 0$, and $X_n = X_{n-1} + Z_n$ for $n \geq 1$.

(a) Is X_n an independent increment process? Justify your answer.

(b) Is X_n a Markov process? Justify your answer.

(c) Is X_n a Gaussian process? Justify your answer.

(d) Find the mean and autocorrelation functions of X_n.

(e) Specify the first and second order pdfs of X_n.

(f) Specify the joint pdf of X_1, X_2, and X_3.

(g) Find $E(X_{20}|X_1, X_2, \ldots, X_{10})$.

Solution:

(a) Yes. The increments X_{n_1}, $X_{n_2} - X_{n_1}, \ldots, X_{n_{k_1}} - X_{n_k}$ are sums of different Z_is, and the Z_is are IID.

(b) Yes. Since the process has independent increments, it is Markov.

(c) Yes. Any set of samples of X_n, $n \geq 1$ are obtained by a linear transformation of IID $\mathcal{N}(0,1)$ random variables and therefore all nth order distributions of X_n are jointly Gaussian (it is not sufficient to show that the random variable X_n is Gaussian).

(d)

$$E[X_n] = E\left[\sum_{i=0}^{n} Z_i\right] = \sum_{i=0}^{n} E[Z_i] = \sum_{i=0}^{n} 0 = 0.$$

$$\begin{aligned}
R_X(n_1, n_2) &= E[X_{n_1} X_{n_2}] \\
&= E\left[\sum_{i=1}^{n_1} Z_i \sum_{j=1}^{n_2} Z_j\right] \\
&= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} E[Z_i Z_j] \\
&= \sum_{i=1}^{\min(n_1,n_2)} E Z_i^2 \quad \text{(IID)} \\
&= \min(n_1, n_2).
\end{aligned}$$

(e) As shown above, X_n is Gaussian with mean zero and variance

$$\begin{aligned}
\mathrm{Var}(X_n) &= E[X_n^2] - E^2[X_n] \\
&= R_X(n, n) - 0 \\
&= n.
\end{aligned}$$

Thus, $X_n \sim \mathcal{N}(0, n)$.

(f) X_n, $n \geq 1$ is a zero mean Gaussian random process. Thus

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} E[X_1] \\ E[X_2] \\ E[X_3] \end{bmatrix}, \begin{bmatrix} R_X(1,1) & R_X(1,2) & R_X(1,3) \\ R_X(2,1) & R_X(2,2) & R_X(2,3) \\ R_X(3,1) & R_X(3,2) & R_X(3,3) \end{bmatrix} \right)$$

Substituting, we get

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \right)$$

(g) Since X_n is an independent increment process,

$$\begin{aligned} E(X_{20}|X_1, X_2, \ldots, X_{10}) &= E(X_{20} - X_{10} + X_{10}|X_1, X_2, \ldots, X_{10}) \\ &= E(X_{20} - X_{10}|X_1, X_2, \ldots, X_{10}) + E(X_{10}|X_1, X_2, \ldots, X_{10}) \\ &= E(X_{20} - X_{10}) + X_{10} \\ &= E(X_{10}) + X_{10} \\ &= 0 + X_{10} \\ &= X_{10}. \end{aligned}$$

6. *Random binary waveform.* In a digital communication channel the symbol "1" is represented by the fixed duration rectangular pulse

$$g(t) = \begin{cases} 1 & \text{for } 0 \leq t < 1 \\ 0 & \text{otherwise,} \end{cases}$$

and the symbol "0" is represented by $-g(t)$. The data transmitted over the channel is represented by the random process

$$X(t) = \sum_{k=0}^{\infty} A_k g(t-k), \quad \text{for } t \geq 0,$$

where $A_0, A_1, \ldots$ are i.i.d random variables with

$$A_i = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2}. \end{cases}$$

(a) Find its first and second order pmfs.

(b) Find the mean and the autocorrelation function of the process $X(t)$.

Solution:

(a) The first order pmf is:

$$\begin{aligned}
p_{X(t)}(x) &= \mathrm{P}\left(X\left(t\right) = x\right) \\
&= \mathrm{P}\left(\sum_{k=0}^{\infty} A_k g(t-k) = x\right) \\
&= \mathrm{P}\left(A_{\lfloor t \rfloor} = x\right) \\
&= \mathrm{P}\left(A_0 = x\right) \quad \text{IID} \\
&= \begin{cases} \frac{1}{2}, & x = \pm 1 \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Now note that $X(t_1)$ and $X(t_2)$ are dependent only if t_1 and t_2 fall within the same time interval. Otherwise, they are independent. Thus, the second order pmf is:

$$\begin{aligned}
p_{X(t_1)X(t_2)}(x,y) &= \mathrm{P}\left(X\left(t_1\right) = x, X\left(t_2\right) = y\right) \\
&= \mathrm{P}\left(\sum_{k=0}^{\infty} A_k g\left(t_1 - k\right) = x, \sum_{k=0}^{\infty} A_k g\left(t_2 - k\right) = y\right) \\
&= \mathrm{P}\left(A_{\lfloor t_1 \rfloor} = x, A_{\lfloor t_2 \rfloor} = y\right) \\
&= \begin{cases} \mathrm{P}(A_0 = x, A_0 = y), & \lfloor t_1 \rfloor = \lfloor t_2 \rfloor \\ \mathrm{P}(A_0 = x, A_1 = y), & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{2}, & \lfloor t_1 \rfloor = \lfloor t_2 \rfloor \ \&\ (x,y) = (1,1), (-1,-1) \\ \frac{1}{4}, & \lfloor t_1 \rfloor \neq \lfloor t_2 \rfloor \ \&\ (x,y) = (1,1), (1,-1), (-1,1), (-1,-1) \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

(b) For $t \geq 0$,

$$\begin{aligned}
E[X(t)] &= E\left[\sum_{k=0}^{\infty} A_k g(t-k)\right] \\
&= \sum_{k=0}^{\infty} g(t-k) E[A_k] \\
&= 0.
\end{aligned}$$

For the autocorrelation $R_X(t_1, t_2)$, we note once again that only if t_1 and t_2 fall within the same interval, will $X(t_1)$ be dependent on $X(t_2)$; if they do not fall in the same interval then they are independent from one another. Then,

$$\begin{aligned}
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\
&= \sum_{k=0}^{\infty} g(t_1 - k) g(t_2 - k) E[A_k^2] \\
&= \begin{cases} 1, & \lfloor t_1 \rfloor = \lfloor t_2 \rfloor \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

7. *Amplitude modulation.* Consider the random process $X(t) = A(t)\cos(\omega t + \Theta)$, where $A(t)$ is a zero-mean WSS process with autocorrelation function $R_A(\tau) = e^{-\frac{1}{2}|\tau|}$, $\Theta \sim \text{Unif}[0, 2\pi]$, and $A(t)$ and Θ are independent. Is $X(t)$ wide sense stationary?

   Solution: $X(t)$ is wide-sense stationary if $EX(t)$ is independent of t and if $R_X(t_1, t_2)$ depends only on $t_1 - t_2$. Consider

$$\begin{aligned} E[X(t)] &= E[A(t)\cos(\omega t + \Theta)] \\ &= E[A(t)]E[\cos(\omega t + \Theta)] \quad \text{by independence} \\ &= 0, \end{aligned}$$

   and

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= E[A(t_1)\cos(\omega t_1 + \Theta)A(t_2)\cos(\omega t_2 + \Theta)] \\ &= E[A(t_1)A(t_2)\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)] \\ &= E[A(t_1)A(t_2) \cdot E\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)] \quad \text{by independence} \\ &= R_A(t_1 - t_2)E[\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)] \\ &= R_A(t_1 - t_2)E\left[\frac{1}{2}\left(\cos(\omega(t_1 + t_2) + 2\Theta) + \cos(\omega(t_1 - t_2))\right)\right] \\ &= \frac{1}{2}R_A(t_1 - t_2)E\begin{pmatrix} \cos(\omega(t_1 + t_2))\cos(2\Theta) \\ -\sin(\omega(t_1 + t_2))\sin(2\Theta) \\ +\cos(\omega(t_1 - t_2)) \end{pmatrix} \\ &= \frac{1}{2}R_A(t_1 - t_2)\begin{pmatrix} E\cos(\omega(t_1 + t_2)) \cdot E\cos(2\Theta) \\ -E\sin(\omega(t_1 + t_2)) \cdot E\sin(2\Theta) \\ +E\cos(\omega(t_1 - t_2)) \end{pmatrix} \\ &= \frac{1}{2}R_A(t_1 - t_2)\cos(\omega(t_1 - t_2)) \\ &= \frac{1}{2}R_A(t_1 - t_2)\cos(\omega(t_1 - t_2)), \end{aligned}$$

   which is a function of $t_1 - t_2$ only. Hence $X(t)$ is wide-sense stationary.

8. *LTI system with WSS process input.* Let $Y(t) = h(t) * X(t)$ and $Z(t) = X(t) - Y(t)$ as shown in the Figure 1.

   (a) Find $S_Z(f)$.

   (b) Find $E(Z^2(t))$.

   Your answers should be in terms of $S_X(f)$ and the transfer function $H(f) = \mathcal{F}[h(t)]$.

   Solution:

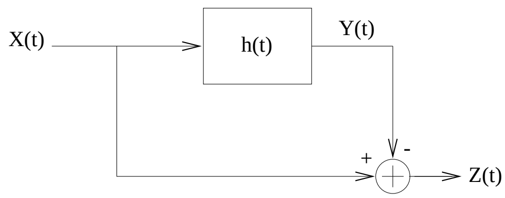

Figure 1: LTI system.

(a) To find $S_Z(f)$, we first find the autocorrelation function

$$\begin{aligned} R_Z(\tau) &= E(Z(t)Z(t+\tau)) \\ &= E((X(t)-Y(t))(X(t+\tau)-Y(t+\tau))) \\ &= R_X(\tau)+R_Y(\tau)-R_{YX}(-\tau)-R_{XY}(-\tau) \\ &= R_X(\tau)+R_Y(\tau)-R_{XY}(\tau)-R_{XY}(-\tau). \end{aligned}$$

Now, taking the Fourier Transform, we get

$$\begin{aligned} S_Z(f) &= S_X(f)+S_Y(f)-S_{XY}(f)-S_{XY}(-f) \\ &= S_X(f)+|H(f)|^2S_X(f)-H(-f)S_X(f)-H(f)S_X(f) \\ &= S_X(f)\left(1+|H(f)|^2-2\text{Re}[H(f)]\right) \\ &= S_X(f)|1-H(f)|^2. \end{aligned}$$

(b) To find the average power of $Z(t)$, we find the area under $S_Z(f)$

$$E(Z^2(t)) = \int_{-\infty}^{\infty} |1-H(f)|^2 S_X(f)\, df.$$

9. *Echo filtering (from Spring 2006 Final).* A signal $X(t)$ and its echo arrive at the receiver as $Y(t)=X(t)+X(t-\Delta)+Z(t)$. Here the signal $X(t)$ is a zero-mean WSS process with power spectral density $S_X(f)$ and the noise $Z(t)$ is a zero-mean WSS with power spectral density $S_Z(f)=N_0/2$, uncorrelated with $X(t)$.

   (a) Find $S_Y(f)$ in terms of $S_X(f)$, Δ, and N_0.

   (b) Find the best linear filter to estimate $X(t)$ from $\{Y(s)\}_{-\infty<s<\infty}$.

   Solution:

   (a) We can write $Y(t)=g(t)*X(t)+Z(t)$ where $g(t)=\delta(t)+\delta(t-\Delta)$. Thus, $S_Y(f)=|G(f)|^2S_X(f)+S_Z(f)=|1+e^{-j2\pi\Delta f}|^2S_X(f)+\frac{N_0}{2}$.

   (b) Since $S_{YX}(f)=(1+e^{-j2\pi\Delta f})S_X(f)$,

$$\hat{X}(t)=h(t)*Y(t),$$

where the linear filter $h(t)$ has the transfer function

$$H(f) = \frac{S_{YX}(f)}{S_Y(f)} = \frac{(1 + e^{-j2\pi\Delta f})S_X(f)}{|1 + e^{-j2\pi\Delta f}|^2 S_X(f) + \frac{N_0}{2}}.$$