Midterm Examination
(Total: 120 points)

There are 3 problems, each problem with 4 parts, each part worth 10 points.

Your answer should be as clear and readable as possible. In particular, if the answer involves a pmf or pdf, make sure to identify the values or intervals for which the pmf or pdf is nonzero.

1. *Coin with random bias.*
   A random variable P is drawn uniformly from the interval $[0, 1]$. Then a coin with bias P is flipped three times. Assume that the value of the bias does not change during the sequence of tosses.

   (a) What is the probability that all three flips are heads?

   (b) Find the probability that the second flip is heads given that the first flip is heads.

   (c) Is the second flip independent of the first flip?

   (d) What is the conditional pdf of the bias P given the first flip is heads?

2. *Two independent uniform random variables.*
   Let X and Y be independently and uniformly drawn from the inverval $[0, 1]$.

   (a) Find the pdf of $U = \max(X, Y)$.

   (b) Find the pdf of $V = \min(X, Y)$.

   (c) Find the pdf of $W = U - V$.
   (Hint: In case you are stuck, you can start working on part (d) first.)

   (d) Find the probability $\mathrm{P}\{|X - Y| \geq 1/2\}$.

3. *One-bit quantization of Gaussian sources.*
Let $X \sim N(0,1)$ and let

$$Y = \begin{cases} 1, & \text{if } X \geq 0, \\ -1, & \text{otherwise.} \end{cases}$$

Thus Y encodes the sign of X.

(a) Find the pmf of Y.

(b) What is the conditional pdf of X given the observation that X is nonnegative? In other words, find $f_{X|Y}(x|1)$.

(c) Find the minimum MSE (mean squared error) estimator of X given Y. That is, find the estimator $g(y)$ that minimizes the MSE

$$E\big[(X - g(Y))^2\big].$$

(d) What is the associated MSE?