

Solutions to Midterm Examination
(Total:120 points)

There are 3 problems, each with 4 parts. Each part is 10 points.

Your answers should be as clear and readable as possible. In particular, if your answer involves a pdf or pmf, make sure to identify the values or intervals for which the pdf or pmf is nonzero.

1. *Coin with random bias.*

Let P be a random variable distributed uniformly over $[0, 1]$. A coin with (random) bias P is flipped three times. Assume that the value of the bias does not change during the sequence of tosses.

- (a) What is the probability that all three flips are heads?
- (b) Find the probability that the second flip is heads given that the first flip is heads.
- (c) Is the second flip independent of the first flip?
- (d) What is the conditional pdf of the random bias P given the first flip is heads?

Solution: Let X_i , $i = 1, 2, 3$, denote the outcome of the i -th coin flip.

- (a) By the law of total probability

$$\begin{aligned} P\{X_1 = H, X_2 = H, X_3 = H\} &= \int_0^1 P\{X_1 = H, X_2 = H, X_3 = H | P = p\} f_P(p) dp \\ &= \int_0^1 p^3 f_P(p) dp \\ &= \int_0^1 p^3 dp \\ &= \frac{1}{4}. \end{aligned}$$

(b) By the definition of conditional probability,

$$P\{X_2 = H|X_1 = H\} = \frac{P\{X_2 = H, X_1 = H\}}{P\{X_1 = H\}}.$$

Again by the law of total probability

$$\begin{aligned} P\{X_1 = H\} &= \int_0^1 P\{X_1 = H|P = p\}f_P(p)dp \\ &= \int_0^1 p dp \\ &= 1/2, \end{aligned}$$

and

$$\begin{aligned} P\{X_1 = H, X_2 = H\} &= \int_0^1 P\{X_1 = H, X_2 = H|P = p\}f_P(p)dp \\ &= \int_0^1 p^2 dp \\ &= 1/3. \end{aligned}$$

Hence,

$$P\{X_2 = H|X_1 = H\} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

(c) First note that

$$P\{X_2 = H\} = P\{X_1 = H\} = 1/2.$$

(Why?) On the other hand, we have already found

$$P\{X_2 = H|X_1 = H\} = \frac{2}{3} \neq P\{X_2 = H\}.$$

Hence, X_1 and X_2 are not independent.

(d) By the Bayes rule, we have

$$\begin{aligned} f_{P|X_1}(p|H) &= \frac{p_{X_1|P}(H|p)f_P(p)}{p_{X_1}(H)} \\ &= \begin{cases} \frac{p \cdot 1}{1/2} = 2p, & 0 \leq p \leq 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

2. *Two independent uniform random variables.*

Let X and Y be independently and uniformly drawn from the interval $[0, 1]$.

- (a) Find the pdf of $U = \max(X, Y)$.
- (b) Find the pdf of $V = \min(X, Y)$.
- (c) Find the pdf of $W = U - V$.
- (d) Find the probability $P\{|X - Y| \geq 1/2\}$.

Solution:

(a) We have

$$\begin{aligned} F_U(u) &= P\{U \leq u\} \\ &= P\{\min(X, Y) \leq u\} \\ &= P\{X \leq u, Y \leq u\} \\ &= P\{X \leq u\}P\{Y \leq u\} \\ &= u^2 \end{aligned}$$

for $0 \leq u \leq 1$. Hence,

$$f_U(u) = \begin{cases} 2u, & 0 \leq u \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Similarly,

$$\begin{aligned} 1 - F_V(v) &= P\{V > v\} \\ &= P\{\max(X, Y) > v\} \\ &= P\{X > v, Y > v\} \\ &= P\{X > v\}P\{Y > v\} \\ &= (1 - v)^2, \end{aligned}$$

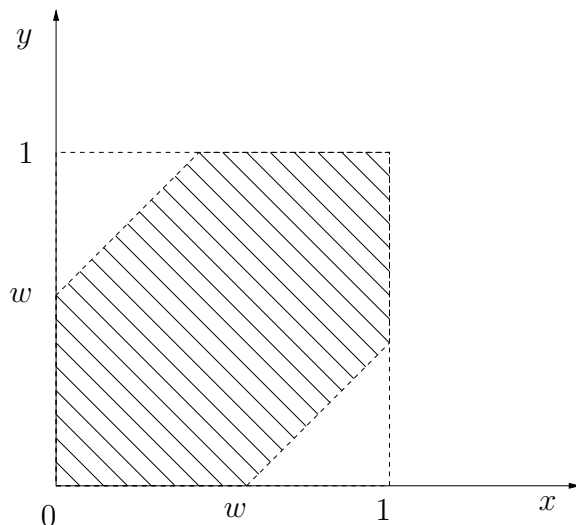
or equivalently, $F_V(v) = 1 - (1 - v)^2$, for $0 \leq v \leq 1$. Hence,

$$f_V(v) = \begin{cases} 2(1 - v), & 0 \leq v \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(c) First note that $W = U - V = |X - Y|$. (Why?) Hence,

$$\begin{aligned} P\{W \leq w\} &= P\{|X - Y| \leq w\} \\ &= P(-w \leq X - Y \leq w). \end{aligned}$$

Since X and Y are uniformly distributed over $[0, 1]$, the above integral is equal to the area of the shaded region in the following figure:



The area can be easily calculated as $1 - (1 - w)^2$ for $0 \leq w \leq 1$. Hence $F_W(w) = 1 - (1 - w)^2$ and

$$f_W(w) = \begin{cases} 2(1 - w), & 0 \leq w \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

We can alternatively calculate $f_W(w)$ following the method presented in Question 7 in Homework Set #8.

(d) From the figure above,

$$\mathbb{P}\{|X - Y| \geq 1/2\} = \mathbb{P}\{W \geq 1/2\} = 1/4.$$

3. *One-bit quantization of Gaussian sources.*

Let $X \sim N(0, 1)$ and let

$$Y = \begin{cases} 1, & \text{if } X \geq 0, \\ -1, & \text{otherwise.} \end{cases}$$

Thus Y encodes the sign of X .

- (a) Find the pmf of Y .
- (b) Find the conditional pdf of X given the observation that X is positive. In other words, find $f_{X|Y}(x|1)$.
- (c) Find the minimum MSE (mean squared error) estimator of X given Y . That is, find the estimator $g(y)$ that minimizes the MSE

$$E[(X - g(Y))^2].$$

- (d) What is the associated MSE?

Solution:

(a) Since $X \sim N(0, 1)$,

$$\begin{aligned}P\{Y = 1\} &= P\{X > 0\} = \frac{1}{2}, \\P\{Y = -1\} &= P\{X < 0\} = \frac{1}{2}.\end{aligned}$$

Therefore, the pmf of Y is given by

$$p_Y(y) = \begin{cases} \frac{1}{2}, & \text{if } y = 1 \text{ or } y = -1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) We first find the conditional cdf of X given $\{Y = 1\}$ and differentiate it to find the pdf. For $x \geq 0$,

$$\begin{aligned}F_{X|Y}(x|1) &= P\{X \leq x|Y = 1\} \\&= P\{X \leq x|X \geq 0\} \\&= \frac{P\{0 \leq X \leq x\}}{P\{X \geq 0\}} \\&= \frac{F_X(x) - F_X(0)}{F_X(0)} \\&= \frac{F_X(x) - 1/2}{1/2} \\&= 2F_X(x) - 1.\end{aligned}$$

Hence,

$$\begin{aligned}f_{X|Y}(x|1) &= \begin{cases} 2f_X(x), & x > 0, \\ 0, & \text{otherwise} \end{cases} \\&= \begin{cases} \sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

(c) The MMSE estimate of X given Y is $E[X|Y]$. Since $[X|Y = y] = \int_0^1 x f_{X|Y}(x|y) dx$,

$$\begin{aligned}E[X|Y = 1] &= \int_{-\infty}^{\infty} x f_{X|Y}(x|1) dx \\&= \int_0^{\infty} x \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} dx \\&= \sqrt{\frac{2}{\pi}}.\end{aligned}$$

Similarly, we have

$$\begin{aligned} E[X|Y = -1] &= \int_{-\infty}^{\infty} x f_{X|Y}(x| -1) dx \\ &= \int_{-\infty}^0 x \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} dx \\ &= -\sqrt{\frac{2}{\pi}}. \end{aligned}$$

In summary, the MMSE estimate $g^*(Y)$ is given by

$$g^*(Y) = E[X|Y] = \begin{cases} \sqrt{\frac{2}{\pi}}, & Y = 1, \\ -\sqrt{\frac{2}{\pi}}, & Y = -1. \end{cases}$$

(d) The MSE is given by

$$\begin{aligned} E(\text{Var}(X|Y)) &= EX^2 - E((E(X|Y))^2) \\ &= 1 - \left[\frac{1}{2} \cdot \left(\sqrt{\frac{2}{\pi}} \right)^2 + \frac{1}{2} \cdot \left(-\sqrt{\frac{2}{\pi}} \right)^2 \right] \\ &= 1 - \frac{2}{\pi} \\ &= \frac{\pi - 2}{\pi}. \end{aligned}$$