UCSD ECE 153 Handout #22
Prof. Young-Han Kim Tuesday, December 2, 2008

Old Final Exam (Spring 2007)

(Total: 120 points)

1. *Breaking a stick (20 points).* Take a stick of length 1. Break it into two pieces at a location chosen uniformly ar random. Throw away the longer piece and take the shorter one, whose length will be denoted by a random variable X.

   (a) *(5 points)* What is the expected length EX of the remaining stick?
   (Hint: Let U denote the uniform location of break. Then $X = \min(U, 1 - U)$.)

   (b) *(5 points)* Find the pdf of X.

   Break the remaining stick once again and take the shorter piece of length Y.

   (c) *(5 points)* What is the expected length EY of the remaining stick?

   (d) *(5 points)* Find the MMSE estimate of X given Y.
   (Hint: Find the conditional pdf $f_{X|Y}(x|y)$ first. Recall that $\int \frac{1}{t}\,dt = \ln t + c$.)

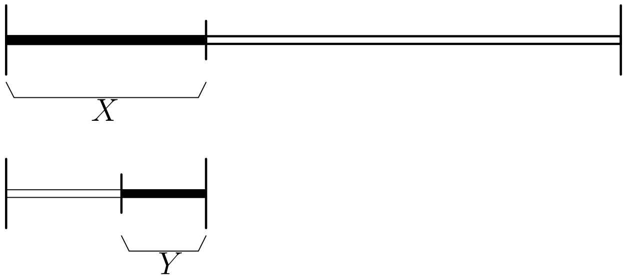

Figure 1: Breaking a stick.

2. *Iocane or Sennari: Return of the chemistry professor (20 points).* An absent-minded chemistry professor forgets to label two identically looking bottles. One contains a chemical named "Iocane" and the other contains a chemical named "Sennari". It is well known that the radioactivity level of "Iocane" has the Unif$[0, 1]$ distribution, while the radioactivity level of "Sennari" has the Exp(1) distribution.

   In the midterm, we found the optimal ule to decide which bottle is which, by measuring the radioactivity level of one of the bottles. The chemistry professor got smarter this time; she now measures both bottles.

   (a) *(10 points)* Let X be the radioactivity level measured from one bottle, and let Y be the radioactivity level measured from the other bottle. What is the optimal decision rule (based on the measurement (X, Y)) that maximizes the chance of correctly identifying the contents? Assume that the radioactivity level of one chemical is independent of the level of the other bottle (conditioned on which bottle contains which).

(b) *(10 points)* What is the associated probability of error?
(Hint: Recall that $\int te^{-t}dt = -(t+1)e^{-t} + c$.)

3. *Fading channel (20 points).* Consider the channel in Figure 2, where the signal X, the multiplicative noise H, and the additive noise Z are independent. Assume that $EX = EZ = 0$ and $EH = 1$.

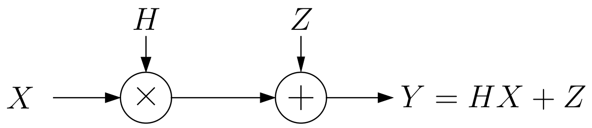

Figure 2: Channel with additive and multiplicative noise.

(a) *(10 points)* Find the MMSE linear estimate of X given Y in terms of σ_H^2, σ_X^2, σ_Z^2, and Y.

(b) *(5 points)* Suppose there is no multiplicative noise, i.e., $H \equiv 1$ is no longer random. Find the MMSE linear estimate of X given Y in terms of σ_X^2, σ_Z^2, and Y.

(c) *(5 points)* Do you prefer H to be random or deterministic? Justify your answer.

4. *Echo filtering (20 points).* A signal $X(t)$ and its echo arrive at the receiver $Y(t)$, where $Y(t) = X(t) + X(t-\Delta) + Z(t)$. The signal $X(t)$ is WSS with zero mean and has power spectral density $S_X(f)$. The noise $Z(t)$ is WSS with zero mean and power spectral density $S_Z(f) = N_0/2$, and is independent of $X(t)$.

(a) *(10 points)* Find $S_Y(f)$.

(b) *(10 points)* Find the best linear filter to estimate $X(t)$ from $\{Y(s)\}_{-\infty<s<\infty}$.

5. *Arrow of time (40 points).* Let X_0 be a Gaussian random variable with zero mean and unit variance, and $X_n = \alpha X_{n-1} + Z_n$ for $n \geq 1$, where α is a fixed constant with $|\alpha| < 1$ and $Z_1, Z_2, \ldots$ are i.i.d. $\sim N(0, 1-\alpha^2)$, independent of X_0.

(a) *(5 points)* Is the process $\{X_n\}$ Gaussian?

(b) *(5 points)* Is $\{X_n\}$ Markov?

(c) *(5 points)* Is $\{X_n\}$ wide sense stationary?

(d) *(5 points)* Is $\{X_n\}$ strict sense stationary?

(e) *(5 points)* Find $R_X(n, m)$.

(f) *(5 points)* Find the (nonlinear) MMSE estimate of X_{100} given $(X_1, X_2, \ldots, X_{99})$.

(g) *(5 points)* Find the MMSE estimate of X_{100} given $(X_{101}, X_{102}, \ldots, X_{199})$.

(h) *(5 points)* Find the MMSE estimate of X_{100} given $(X_1, \ldots, X_{99}, X_{101}, \ldots, X_{199})$.