

Old Final Exam II (Spring 2008)

(Total: 150 points)

1. *Coin with random bias (20 points)*. You are given a coin but are not told what its bias (probability of heads) is. You are told instead that the bias is the outcome of a random variable $P \sim \text{Unif}[0, 1]$. Assume P does not change during the sequence of tosses.

- (a) What is the probability that the first three flips are heads?
- (b) What is the probability that the second flip is heads given that the first flip is heads?

2. *Estimation (20 points)*. Let X_1 and X_2 be independent identically distributed random variables. Let $Y = X_1 + X_2$.

- (a) Find $E[X_1 - X_2|Y]$.
- (b) Find the minimum mean squared error estimate of X_1 given an observed value of $Y = X_1 + X_2$. (Hint: Consider $E[X_1 + X_2|X_1 + X_2]$.)

3. *Stationary process (20 points)*. Consider the Gaussian autoregressive random process

$$X_{k+1} = \frac{1}{3}X_k + Z_k, \quad k = 0, 1, 2, \dots,$$

where Z_0, Z_1, Z_2, \dots are i.i.d. $\sim N(0, 1)$.

- (a) Find the distribution on X_0 that makes this a stationary stochastic process.
- (b) What is the resulting autocorrelation $R_X(n)$?

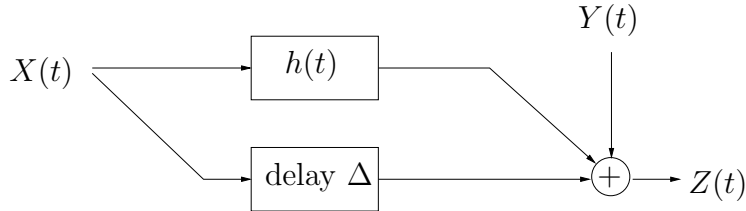
4. *Prediction error process (20 points)*. Let X_1, X_2, \dots be a discrete-time random process. Let \hat{X}_n denote the linear MMSE estimator of X_n given X_1, X_2, \dots, X_{n-1} . Thus \hat{X}_n is of the form $\hat{X}_n = \sum_{i=1}^{n-1} a_{ni}X_i + b_n$. Let $Z_n = X_n - \hat{X}_n$ denote the error.

- (a) Find EZ_n .
- (b) Find $\text{Cov}(Z_n, Z_m)$ for $m > n$. Interpret.

5. *Minimum power (30 points)*. Let $X(t)$ and $Y(t)$ be jointly WSS processes with zero mean. Let

$$\begin{aligned} Z(t) &= h(t) * X(t) + X(t - \Delta) + Y(t) \\ &= \int_{-\infty}^{\infty} h(s)X(t - s)ds + X(t - \Delta) + Y(t) \end{aligned}$$

be the output of the LTI system shown below.



- (a) Find the power spectral density $S_Z(f)$ of the output $Z(t)$. Your answer should be in terms of $S_X(f)$, $S_Y(f)$, $S_{XY}(f)$, Δ , and $H(f)$.
- (b) Now find the LTI filter $H(f)$ that minimizes the power of $Z(t)$. (Hint: Consider $U(t) = -X(t - \Delta) - Y(t)$. Which filter minimizes the power of $Z(t) = h(t) * X(t) - U(t)$?)
- (c) What is the resulting power $EZ^2(t)$?
6. *Integration (40 points)*. Let $X(t)$ be a zero-mean WSS process with autocorrelation function $R_X(\tau) = e^{-|\tau|}$. Let

$$Y(t) = \int_t^{t+1} X(s)ds.$$

- (a) Is $Y(t)$ WSS?
- (b) Is $(X(t), Y(t))$ jointly WSS?
- (c) Find the linear MMSE estimate of $Y(t)$ given $X(t)$. Leave your answer in terms of e .
- (d) Find the linear MMSE estimate of $Y(t)$ given $X(t)$ and $X(t + 1)$.