Solutions to Old Final Exam

1. *Breaking a stick (20 points).*
Take a stick of length 1. Break it at a location chosen uniformly ar random. Throw away the longer part and take the shorter one, the length of which will be denoted by a random variable X.

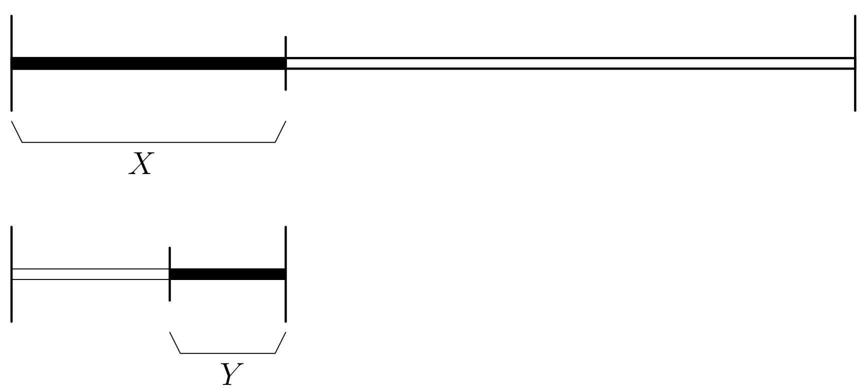
   (a) *(5 points)* What is the expected length EX of the remaining stick?
   (Hint: Let U denote the uniform location of break. Then $X = \min(U, 1-U)$.)

   (b) *(5 points)* Find the pdf of X.

   Break the remaining stick once again and take the shorter one of length Y.

   (c) *(5 points)* What is the expected length EY of the remaining stick?

   (d) *(5 points)* Find the MMSE estimate of X given Y.
   (Hint: Find the conditional pdf $f_{X|Y}(x|y)$ first. Recall that $\int \frac{1}{t}\, dt = \ln t + c$.)



Solution: First note that X cannot be larger than 1/2. From the provided hint, for $0 \le x \le 1/2$,

$$\Pr(X \ge x) = \Pr(x \le U \le 1-x) = 1 - 2x,$$

or equivalently, $\Pr(X \le x) = 2x$, so that X is $\sim \mathrm{U}(0, 1/2)$.

(a) $EX = 1/4$.

(b)

$$f_X(x) = \begin{cases} 2, & 0 \le x \le 1/2, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Similarly, conditioned on $\{X = x\}$, Y is a uniform r.v. on $[0, x/2]$. Thus, $E(Y|X) = X/4$, and

$$EY = EE(Y|X) = E(X/4) = 1/16.$$

(d) First note that for $0 \le y \le x/2 \le 1/4$,

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x) f_{Y|X}(y|x) \\ &= 2 \cdot \frac{2}{x} \\ &= \frac{4}{x}. \end{aligned}$$

Thus, for $0 \le y \le x/2 \le 1/4$,

$$f_Y(y) = \int f_{X,Y}(x,y)dx = \int_{2y}^{\frac{1}{2}} \frac{4}{x} dx = 4(\ln(1/2) - \ln(2y)) = 4\ln(1/4y),$$

and

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1}{x\ln(1/4y)}.$$

Therfore,

$$\begin{aligned} E(X|Y=y) &= \int_{2y}^{\frac{1}{2}} x f_{X|Y}(x|y) dx \\ &= \int_{2y}^{\frac{1}{2}} \frac{1}{\ln(1/4y)} dx \\ &= \frac{1/2 - 2y}{\ln(1/2) - \ln(2y)}, \end{aligned}$$

and

$$E(X|Y) = \frac{1/2 - 2Y}{\ln(1/2) - \ln(2Y)}.$$

2. *Iocane or Sennari: Return of the chemistry professor (20 points).*
An absent-minded chemistry professor forgets to label two identically looking bottles. One contains a chemical named “Iocane” and the other contains a chemical named

"Sennari". It is well known that the radioactivity level of "Iocane" has the Unif$[0, 1]$ distribution, while the radioactivity level of "Sennari" has the Exp(1) distribution.

In the midterm, we found the optimal decision rule to find out which bottle is which by measuring the radioactivity level of one of the bottles. The chemistry professor got smarter this time; she now measures both bottles.

(a) *(10 points)* Let X be the radioactivity level measured from one bottle, and let Y be the radioactivity level measured from the other bottle. What is the optimal decision rule (based on the measurement (X, Y)) that maximizes the chance of correctly identifying the contents? Assume that the radioactivity level of one chemical is independent of that of the other.

(b) *(10 points)* What is the associated probability of error?
(Hint: Recall that $\int te^t dt = -(t+1)e^{-t} + c$.)

Solution: Let $\Theta = 0$ denote the case in which the first bottle (measurement X) is "Iocane" and the second bottle (measurement Y) is "Sennari". Let $\Theta = 1$ denote the other case.

(a) The optimal MAP rule is equivalent to the ML rule

$$D(x, y) = \begin{cases} 0, & f_{X,Y|\Theta}(x, y|0) > f_{X,Y|\Theta}(x, y|1), \\ 1, & \text{otherwise.} \end{cases}$$

Since $f_{X,Y|\Theta}(x, y|0) = 1_{0 \le x \le 1} \cdot e^{-y}$ and $f_{X,Y|\Theta}(x, y|1) = e^{-x} \cdot 1_{0 \le y \le 1}$,

$$D(x, y) = \begin{cases} 0, & (x < 1, y > 1) \text{ or } (0 \le y < x \le 1), \\ 1, & \text{otherwise.} \end{cases}$$

(b) The probability of error is given by

$$\begin{aligned} \Pr(\Theta \neq D(X, Y)) &= \frac{1}{2}\Pr(0 \le X \le Y \le 1|\Theta = 0) + \frac{1}{2}\Pr(0 \le Y < X \le 1|\Theta = 1) \\ &= \Pr(0 \le X \le Y \le 1|\Theta = 0) \\ &= \int_0^1 \int_0^y e^{-y} dx dy \\ &= 1 - 2e^{-1}, \end{aligned}$$

which is less than the error probability $\frac{1}{2}(1 - e^{-1})$ from the single measurement in midterm.

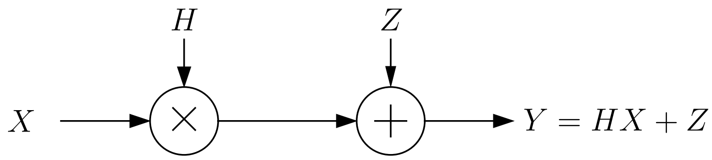


Figure 1: Channel with additive and multiplicative noise.

3. *Fading channel (20 points).*
Consider the channel in Figure 1, where the signal X, the multiplicative noise H, and the additive noise Z are independent. Assume that $EX = EZ = 0$ and $EH = 1$.

   (a) *(10 points)* Find the MMSE linear estimate of X given Y in terms of σ_H^2, σ_X^2, σ_Z^2, and Y.

   (b) *(5 points)* Suppose there is no multiplicative noise, i.e., $H \equiv 1$ is no longer random. Find the MMSE linear estimate of X given Y in terms of σ_X^2, σ_Z^2, and Y.

   (c) *(5 points)* Do you prefer H to be random or deterministic? Justify your answer.

Solution:

(a) The MMSE linear estimate is of the form $\hat{X} = \frac{\mathrm{Cov}(X,Y)}{\sigma_Y^2}(Y - E(Y)) + E(X)$. Now we have $E(X) = 0$, $E(Y) = E(HX) + E(Z) = E(H)E(X) = 0$,

$$\begin{aligned}\sigma_Y^2 &= E((HX+Z)^2)\\ &= E((HX)^2) + E(Z^2)\\ &= E(H^2)E(X^2) + \sigma_Z^2\\ &= (\sigma_H^2 + 1)\sigma_X^2 + \sigma_Z^2,\end{aligned}$$

and $\mathrm{Cov}(X,Y) = E(XY) = E(HX^2) + E(XZ) = \sigma_X^2$. Thus

$$\hat{X} = \frac{\sigma_X^2}{(\sigma_H^2 + 1)\sigma_X^2 + \sigma_Z^2} Y.$$

(b) This case is equivalent to $\sigma_H^2 = 0$. Thus, $\hat{X} = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_Z^2} Y$.

(c) The MSE for part (a) is $\mathrm{MSE}_{\text{random}} = \sigma_X^2 \left(1 - \frac{\sigma_X^2}{(\sigma_H^2+1)\sigma_X^2 + \sigma_Z^2}\right)$, while the MSE for part (b) is $\mathrm{MSE}_{\text{deterministic}} = \sigma_X^2 \left(1 - \frac{\sigma_X^2}{\sigma_X^2 + \sigma_Z^2}\right)$. Thus, $\mathrm{MSE}_{\text{random}} \geq \mathrm{MSE}_{\text{deterministic}}$ and the deterministic H results in a preferably smaller estimation error.

4. *Echo filtering (20 points).*
A signal $X(t)$ and its echo arrive at the receiver $Y(t)$, where

$$Y(t) = X(t) + X(t-\Delta) + Z(t).$$

The signal $X(t)$ is WSS with zero mean and has power spectral density $S_X(f)$. The noise $Z(t)$ is WSS with zero mean and power spectral density $S_Z(f) = N_0/2$, and is independent of $X(t)$.

(a) *(10 points)* Find $S_Y(f)$.

(b) *(10 points)* Find the best linear filter to estimate $X(t)$ from $\{Y(s)\}_{-\infty<s<\infty}$.

Solution:

(a) We can write $Y(t) = g(t) * X(t) + Z(t)$ where $g(t) = \delta(t) + \delta(t-\Delta)$. Thus, $S_Y(f) = |G(f)|^2 S_X(f) + S_Z(f) = |1 + e^{-j2\pi\Delta f}|^2 S_X(f) + \frac{N_0}{2}$.

(b) Since $S_{YX}(f) = (1 + e^{-j2\pi\Delta f}) S_X(f)$,

$$\hat{X}(t) = h(t) * Y(t),$$

where the linear filter $h(t)$ has the transfer function

$$H(f) = \frac{S_{YX}(f)}{S_Y(f)} = \frac{(1 + e^{-j2\pi\Delta f}) S_X(f)}{|1 + e^{-j2\pi\Delta f}|^2 S_X(f) + \frac{N_0}{2}}.$$

5. *Arrow of time (40 points).*
Let X_0 be a Gaussian random variable with zero mean and unit variance, and $X_n = \alpha X_{n-1} + Z_n$ for $n \geq 1$, where $-1 < \alpha < 1$ is a fixed constant and $Z_1, Z_2, \ldots$ are i.i.d. $\sim N(0, 1-\alpha^2)$, independent of X_0.

(a) *(5 points)* Is the process Gaussian?

(b) *(5 points)* Is the process Markov?

(c) *(5 points)* Is the process wide sense stationary?

(d) *(5 points)* Is the process strict sense stationary?

(e) *(5 points)* Find $R_X(n, m)$.

(f) *(5 points)* Find the (nonlinear) MMSE estimate of X_{100} given $(X_1, X_2, \ldots, X_{99})$.

(g) *(5 points)* Find the MMSE estimate of X_{100} given $(X_{101}, X_{102}, \ldots, X_{199})$.

(h) *(5 points)* Find the MMSE estimate of X_{100} given $(X_1, \ldots, X_{99}, X_{101}, \ldots, X_{199})$.

Solution:

(a) Yes, the process is Gaussian, since it is the linear transform of white Gaussian process $\{Z_n\}$.

(b) Yes, the process is Markov since $f(x_{n+1}|x_1,\ldots,x_n) = f(x_{n+1}|x_n)$.

(c) Yes, the process is wide sense stationary, since $EX_n = 0$ for all n and $R_X(n,m)$ is a function of $|n-m|$ only, as we will see in part (e).

(d) Yes, the process is strict sense stationary, since the process is wide sense stationary and Gaussian.

(e) First note that $R_X(n,n) = \alpha^2 R_X(n-1,n-1) + (1-\alpha^2) = 1$ for all n. Since we can express (cf. Homework Set #7, Q5)

$$X_n = \alpha^k X_{n-k} + \alpha^{k-1} Z_{n-k+1} + \cdots + Z_n,$$

and X_{n-k} is independent of $(Z_{n-k+1},\ldots,Z_n)$, we have $R_X(n,n-k) = EX_nX_{n-k} = \alpha^k$. Thus, $R_X(n,m) = \alpha^{|n-m|}$.

(f) Because the process is Gaussian, the MMSE estimator is linear. From Markovity, $\hat{X}_{100} = E(X_{100}|X_1,\ldots,X_{99}) = E(X_{100}|X_{99}) = \frac{R_X(100,99)}{R_X(99,99)} X_{99} = \alpha X_{99}$.

(g) Again from Markovity, $\hat{X}_{100} = E(X_{100}|X_{101},\ldots,X_{199}) = E(X_{100}|X_{101}) = \alpha X_{101}$.

(h) First note that X_{100} is conditionally independent of $(X_1,\ldots,X_{98},X_{102},\ldots,X_{199})$ given (X_{99},X_{101}). To see this, consider

$$\begin{aligned}
&f(x_{100}|x_1,\ldots,x_{99},x_{101},\ldots,x_{199})\\
&\quad= \frac{f(x_1,\ldots,x_{199})}{f(x_1,\ldots,x_{99},x_{101},\ldots,x_{199})}\\
&\quad= \frac{f(x_1)f(x_2|x_1)\cdots f(x_{99}|x_{98})f(x_{100}|x_{99})f(x_{101}|x_{100})f(x_{102}|x_{101})\cdots f(x_{199}|x_{198})}{f(x_1)f(x_2|x_1)\cdots f(x_{99}|x_{98})f(x_{101}|x_{99})f(x_{102}|x_{101})\cdots f(x_{199}|x_{198})}\\
&\quad= \frac{f(x_{100}|x_{99})f(x_{101}|x_{100})}{f(x_{101}|x_{99})}\\
&\quad= \frac{f(x_{100},x_{101}|x_{99})}{f(x_{101}|x_{99})}\\
&\quad= f(x_{100}|x_{99},x_{101}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{X}_{100} &= E(X_{100}|X_{99},X_{101})\\
&= \begin{bmatrix} R_X(100,99) & R_X(100,101) \end{bmatrix} \begin{bmatrix} R_X(99,99) & R_X(99,101) \\ R_X(101,99) & R_X(101,101) \end{bmatrix}^{-1} \begin{bmatrix} X_{99} \\ X_{101} \end{bmatrix}\\
&= \begin{bmatrix} \alpha & \alpha \end{bmatrix} \begin{bmatrix} 1 & \alpha^2 \\ \alpha^2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} X_{99} \\ X_{101} \end{bmatrix}\\
&= \frac{\alpha}{1+\alpha^2}(X_{99} + X_{101}).
\end{aligned}$$