UCSD ECE 153 — Handout #15
Prof. Young-Han Kim — Wednesday, October 29, 2008

Solutions to Old Midterm Exam II

1. *First available teller (20 points).* Consider a bank with two tellers. The service times for the tellers are independent exponentially distributed random variables $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \text{Exp}(\lambda_2)$, respectively. You arrive at the bank and find that both tellers are busy but that nobody else is waiting to be served. You are served by the first available teller once he/she becomes free. Let the random variable Y denote your waiting time. Find the pdf of Y.

   Solution: This problem is very similar to Question 4 in Homework Set #4. First observe that $Y = \min(X_1, X_2)$. Since

$$\begin{aligned}
\text{P}\{Y > y\} &= \text{P}\{X_1 > y, X_2 > y\} \\
&= \text{P}\{X_1 > y\}\text{P}\{X_2 > y\} \\
&= e^{-\lambda_1 y} \times e^{-\lambda_2 y} \\
&= e^{-(\lambda_1+\lambda_2)y}
\end{aligned}$$

   for $y \geq 0$, Y is an exponential random variable with pdf

$$f_Y(y) = \begin{cases} (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)y}, & y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

2. *Sum of packet arrivals (40 points).* Consider a network router with two types of incoming packets, wireline and wireless. Let the random variable $N_1(t)$ denote the number of *wireline* packets arriving during time $(0, t]$ and let the random variable $N_2(t)$ denote the number of *wireless* packets arriving during time $(0, t]$. Suppose $N_1(t)$ and $N_2(t)$ are independent Poisson with pmfs

$$\begin{aligned}
\text{P}\{N_1(t) = n\} &= \frac{(\lambda_1 t)^n}{n!} e^{-\lambda_1 t} \qquad \text{for } n = 0, 1, 2, \ldots \\
\text{P}\{N_2(t) = k\} &= \frac{(\lambda_2 t)^k}{k!} e^{-\lambda_2 t} \qquad \text{for } k = 0, 1, 2, \ldots.
\end{aligned}$$

   Let $N(t) = N_1(t) + N_2(t)$ be the total number of packets arriving at the router during time $(0, t]$.

   (a) Find the mean $E(N(t))$ and variance $\text{Var}(N(t))$ of the total number of packet arrivals.

(b) Find the pmf of $N(t)$.

(c) Let the random variable Y be the time to receive the first packet of either type. Find the pdf of Y.

(d) What is the probability that the first received packet is wireless?

Solution:

(a) Since N_1 and N_2 are independent,

$$EN = EN_1 + EN_2 = \lambda_1 t + \lambda_2 t = (\lambda_1 + \lambda_2)t.$$

and

$$\mathrm{Var}(N) = \mathrm{Var}(N_1) + \mathrm{Var}(N_2) = (\lambda_1 + \lambda_2)t.$$

(b) In fact, as discussed in class, $N(t)$ is Poisson itself. To see this,

$$\begin{aligned}
\mathrm{P}\{N = n\} &= \sum_{k=0}^{n} \mathrm{P}\{N_1 = k\}\mathrm{P}\{N_2 = n-k\} \\
&= \sum_{k=0}^{n} \frac{(\lambda_1 t)^k}{k!} e^{-\lambda_1 t} \frac{(\lambda_2 t)^{n-k}}{(n-k)!} e^{-\lambda_2 t} \\
&= \sum_{k=0}^{n} \binom{n}{k} (\lambda_1 t)^k (\lambda_2 t)^{n-k} \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} \\
&= \frac{((\lambda_1 + \lambda_2)t)^n}{n!} e^{-(\lambda_1+\lambda_2)t}
\end{aligned}$$

for $n = 0, 1, 2, \ldots$.

(c) Since $N(t)$ is Poisson with parameter $(\lambda_1 + \lambda_2)t$, the time to the first packet is exponential with parameter $\lambda_1 + \lambda_2$ (cf. Question 2 in Homework Set #3); also recall Question 8 in Homework Set #2. Therefore,

$$f_Y(y) = \begin{cases} (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)y}, & y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Alternatively, we can let X_1 and X_2 denote the times until first packets of each type and see that $Y = \min(X_1, X_2)$. Now we can use the result in Question 1 of this exam.

(d) If we let X_1 and X_2 denote the times until first packets of each type, the probability that the first packet is wireless is simply $\mathrm{P}\{X_2 < X_1\} = \frac{\lambda_2}{\lambda_1+\lambda_2}$. (Recall Question 4 in Homework Set #3.)

3. *Signal or no signal (40 points).* Consider a communication system that is operated only from time to time. When the communication system is in the "normal" mode (denoted by $M = 1$), it transmits a random signal $S = X$ with

$$X = \begin{cases} +1, & \text{with probability } 1/2, \\ -1, & \text{with probability } 1/2. \end{cases}$$

When the system is in the "idle" mode (denoted by $M = 0$), it does not transmit any signal ($S = 0$). Both normal and idle modes occur with equal probability. Thus

$$S = \begin{cases} X, & \text{with probability } 1/2, \\ 0, & \text{with probability } 1/2. \end{cases}$$

The receiver observes $Y = S + Z$, where the ambient noise $Z \sim \text{Unif}[-1, 1]$ is independent of S.

(a) Find and sketch the conditional pdf $f_{Y|M}(y|1)$ of the receiver observation Y given that the system is in the normal mode.

(b) Find and sketch the conditional pdf $f_{Y|M}(y|0)$ of the receiver observation Y given that the system is in the idle mode.

(c) Find the optimal decoder $d(y)$ for deciding whether the system is normal or idle. Provide the answer in terms of intervals of y.

(d) Find the associated probability of error.

Solution:

(a) If $M = 1$,

$$Y = \begin{cases} 1 + Z, & \text{with probability } 1/2, \\ -1 + Z, & \text{with probability } 1/2. \end{cases}$$

Hence, we have

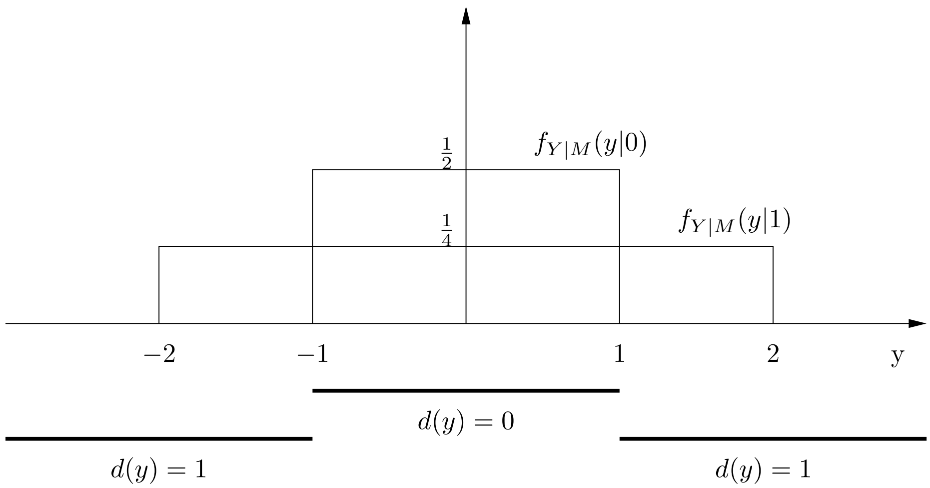
$$f_{Y|M}(y|1) = \begin{cases} \frac{1}{2} f_Z(y-1) + \frac{1}{2} f_Z(y+1) = \frac{1}{4}, & -2 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(b) If $M = 0$, $Y = Z$, so

$$f_{Y|M}(y|0) = \begin{cases} f_Z(y) = \frac{1}{2}, & -1 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Since both modes are equally likely, the optimal MAP decoding rule reduces to the ML rule, in which

$$d(y) = \begin{cases} 0, & \text{if } f_{Y|M}(y|0) > f_{Y|M}(y|1), \\ 1, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 0, & \text{if } -1 < y < 1, \\ 1, & \text{otherwise.} \end{cases}$$



(d) The probability of error is given by

$$\begin{aligned} \mathrm{P}\{M \neq d(Y)\} &= \mathrm{P}\{M = 1, -1 < Y < 1\} \\ &= \mathrm{P}\{M = 1\}\mathrm{P}\{-1 < Y < 1 | M = 1\} \\ &= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}. \end{aligned}$$