UCSD ECE 153 — Handout #14
Prof. Young-Han Kim — Wednesday, October 29, 2008

Solutions to Old Midterm Exam

1. *Light bulbs (20 points).*
   Alice and Bob go shopping for light bulbs. Alice buys two regular light bulbs with life span distributed according to the Exp(1) distribution; she will use these two bulbs one by one. Bob goes for one higher-end bulb with life span distributed according to the Exp(1/2) distribution.

   We will denote X_1 and X_2 for the life spans of Alice's bulbs and denote Y for that of Bob's bulb. We assume that X_1, X_2, and Y are independent of each other.

   (a) *(5 points)* What is the pdf of the total life span of Alice's bulbs, i.e., the pdf of $X_1 + X_2$?

   (b) *(5 points)* Compare the expected life spans of Alice's choice (two cheap bulbs used sequentially) and Bob's choice (one expensive bulb).

   (c) *(10 points)* What is the probability that Bob's bulb will outlive Alice's bulbs?

   You may find the following facts useful.

   - The pdf of an Exp(λ) random variable X is
   $$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$
   - $\displaystyle\int_0^\infty x e^{-\lambda x} dx = \frac{1}{\lambda^2}.$

   Solution:

   (a) Let $Z = X_1 + X_2$. Since X_1 and X_2 are independent,
   $$f_Z(z) = (f_{X_1} * f_{X_2})(z) = \int_0^z e^{-x} e^{-(z-x)} dx = \int_0^z e^{-z} dx = z e^{-z}.$$
   This distribution is called the Gamma distribution.

   (b) $E(X_1 + X_2) = E(X_1) + E(X_2) = 1 + 1 = 2 = E(Y)$.

   (c) This is the probability that $Y \geq X_1 + X_2$. There are many ways to calculate $\mathrm{P}(Y \geq X_1 + X_2)$.

Method I: As in part (a), let $Z = X_1 + X_2$. Since Y and Z are independent, the joint distribution of Y and Z is given by

$$f_{Y,Z}(y,z) = \frac{1}{2}e^{-\frac{1}{2}y} \cdot ze^{-z}$$

for $y, z \geq 0$. Therefore,

$$\begin{aligned}
\mathrm{P}(Y \geq Z) &= \iint_{\{y \geq z\}} f_{Y,Z}(y,z)dydz \\
&= \int_0^\infty \int_z^\infty \frac{1}{2}e^{-\frac{1}{2}y} \cdot ze^{-z}dydz \\
&= \int_0^\infty e^{-\frac{1}{2}z} \cdot ze^{-z}dz \\
&= \int_0^\infty ze^{-\frac{3}{2}z}dz \\
&= \left(\frac{2}{3}\right)^2 \\
&= \frac{4}{9}.
\end{aligned}$$

Method II: We use the law of total probability as follows:

$$\begin{aligned}
P(Y \geq X_1 + X_2) &= \iint P(Y \geq X_1 + X_2 | X_1 = x_1, X_2 = x_2) f_{X_1,X_2}(x_1,x_2)dx_1dx_2 \\
&= \iint P(Y \geq x_1 + x_2 | X_1 = x_1, X_2 = x_2) f_{X_1,X_2}(x_1,x_2)dx_1dx_2 \\
&= \iint P(Y \geq x_1 + x_2) f_{X_1,X_2}(x_1,x_2)dx_1dx_2 \\
&= \int_0^\infty \int_0^\infty e^{-\frac{1}{2}(x_1+x_2)}e^{-x_1}e^{-x_2}dx_1dx_2 \\
&= \left(\int_0^\infty e^{-\frac{3}{2}x}dx\right)^2 \\
&= \left(\frac{2}{3}\right)^2 \\
&= \frac{4}{9}.
\end{aligned}$$

2. *Iocane or Sennari (20 points).*
An absent-minded chemistry professor forgets to label two identically looking bottles. One contains a chemical named "Iocane" and the other contains a chemical named "Sennari". It is well known that the radioactivity level of "Iocane" has the Unif$[0,1]$ distribution, while the radioactivity level of "Sennari" has the Exp(1) distribution.

   (a) *(10 points)* Let X be the radioactivity level measured from one of the bottles. What is the optimal decision rule (based on the measurement X) that maximizes the chance of correctly identifying the content of the bottle?

   (b) *(10 points)* What is the associated probability of error?

Solution: Let $\Theta = 0$ denote the case in which the content of the bottle is "Iocane" and let $\Theta = 1$ denote the case in which the content of the bottle is "Sennari". Implicit in the problem statement is that $\mathrm{P}(\Theta = 0) = \mathrm{P}(\Theta = 1) = 1/2$.

   (a) The optimal MAP rule is equivalent to the ML rule

$$D(x) = \begin{cases} 0, & f_{X|\Theta}(x|0) > f_{X|\Theta}(x|1), \\ 1, & \text{otherwise.} \end{cases}$$

Since the Unif$(0,1)$ pdf $f_{X|\Theta}(x|0)$ is larger than the Exp(1) pdf $f_{X|\Theta}(x|1)$ for $0 < x < 1$, we have

$$D(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & \text{otherwise.} \end{cases}$$

   (b) The probability of error is given by

$$\begin{aligned} \mathrm{P}(\Theta \neq D(X)) &= \frac{1}{2}\mathrm{P}(\Theta \neq D(X)|\Theta = 0) + \frac{1}{2}\mathrm{P}(\Theta \neq D(X)|\Theta = 1) \\ &= \frac{1}{2}\mathrm{P}(X > 1|\Theta = 0) + \frac{1}{2}\mathrm{P}(0 < X < 1|\Theta = 1) \\ &= \frac{1}{2}(1 - e^{-1}). \end{aligned}$$

3. *Binary symmetric channel (40 points).*
The signal X is drawn as

$$X = \begin{cases} +1, & \text{with probability } \frac{1}{2}, \\ -1, & \text{with probability } \frac{1}{2}, \end{cases}$$

and the multiplicative noise Z is *independently* drawn as

$$Z = \begin{cases} +1, & \text{with probability } \frac{3}{4}, \\ -1, & \text{with probability } \frac{1}{4}. \end{cases}$$

Their product $Y = X \cdot Z$ is observed.

(a) *(5 points)* Find the conditional pmf $p_{Y|X}(y|x)$ of Y given X.

(b) *(5 points)* Find the joint pmf $p_{X,Y}(x,y)$ of X and Y.

(c) *(5 points)* Find the marginal pmf $p_Y(y)$ of Y.

(d) *(5 points)* Find the conditional pmf $p_{X|Y}(x|y)$ of X given Y.

(e) *(5 points)* Find the optimal estimator $g(Y)$ that minimize the mean square error $E\left[(X-g(Y))^2\right]$.

(f) *(5 points)* What is the corresponding mean square error?

(g) *(5 points)* Find the optimal decoder $D(Y)$ that minimizes the error probability $\mathrm{P}\{X \neq D(Y)\}$.

(h) *(5 points)* What is the corresponding probability of error?

Solution:

(a)

$$p_{Y|X}(+1|+1) = p_{Y|X}(-1|-1) = 3/4.$$
$$p_{Y|X}(-1|+1) = p_{Y|X}(+1|-1) = 1/4.$$

(b)

$$p_{X,Y}(+1,+1) = p_{X,Y}(-1,-1) = 3/8.$$
$$p_{X,Y}(-1,+1) = p_{X,Y}(+1,-1) = 1/8.$$

(c)

$$p_Y(+1) = p_Y(-1) = 1/2.$$

(d)

$$p_{X|Y}(+1|+1) = p_{X|Y}(-1|-1) = 3/4.$$
$$p_{X|Y}(-1|+1) = p_{X|Y}(+1|-1) = 1/4.$$

(e)

$$g(+1) = E(X|Y=+1) = +1 \times (3/4) + (-1) \times (1/4) = +1/2.$$
$$g(-1) = E(X|Y=-1) = -1 \times (3/4) + (+1) \times (1/4) = -1/2.$$

(f)

$$\begin{aligned} E(X-g(Y))^2 &= \sum_{x,y} (x-g(y))^2 p_{X,Y}(x,y) \\ &= 2 \times (3/8) \times (1/2)^2 + 2 \times (1/8) \times (3/2)^2 = 3/4. \end{aligned}$$

(g) Since $p_{X|Y}(+1|+1) > p_{X|Y}(-1|+1)$ and $p_{X|Y}(-1|-1) > p_{X|Y}(+1|-1)$, $D(Y) = Y$, i.e., $D(+1) = +1$ and $D(-1) = -1$.

(h)

$$\mathrm{P}(X \neq D(Y)) = \mathrm{P}(X \neq Y) = 1/4.$$