A Proof of the Converse for the Capacity of Gaussian MIMO Broadcast Channels

Mehdi Mohseni and John M. Cioffi
Department of Electrical Engineering
Stanford University, Stanford, CA 94305-9510
{mmohseni, cioffi}@stanford.edu

Abstract

The paper provides a proof of the converse for the capacity region of the Gaussian MIMO broadcast channel. The proof uses several ideas from earlier works on the problem including the recent converse proof by Weingarten, Steinberg and Shamai. First the duality between Gaussian multiple access and broadcast channels is used to show that every point on the boundary of the dirty paper coding region can be represented as the optimal solution to a convex optimization problem. Using the optimality conditions for this convex problem, a degraded broadcast channel is constructed for each point. It is then shown that the capacity region for this degraded broadcast channel contains the capacity region of the original channel. Moreover, the same point lies on the boundary of the dirty paper coding region for this degraded channel. Finally, the standard entropy power inequality is used to show that this point lies on the boundary of the capacity region of the degraded channel as well and consequently it is on the boundary of the capacity region of the original channel.

Index Terms–Broadcast channel (BC), multiple access channel (MAC), capacity region, dirty paper coding (DPC) region, duality, convex optimization, Karush-Kuhn-Tucker (KKT) optimality conditions, entropy power inequality (EPI).

1 Introduction

Consider a memoryless Gaussian multiple-input multiple-output (MIMO) broadcast channel (BC) with $K \geq 2$ receivers. Assume that the transmitter has t antennas and each receiver has r antennas. Equal number of receive antennas is chosen to simplify notation. The proof readily applies to the case of receivers with different numbers of antennas. The received symbols of user $k = 1, \ldots, K$ at transmission i can be expressed in terms of the transmitted symbols and channel coefficients as,

$$\mathbf{y}_k(i) = H_k \mathbf{x}(i) + \mathbf{z}_k(i), \tag{1}$$

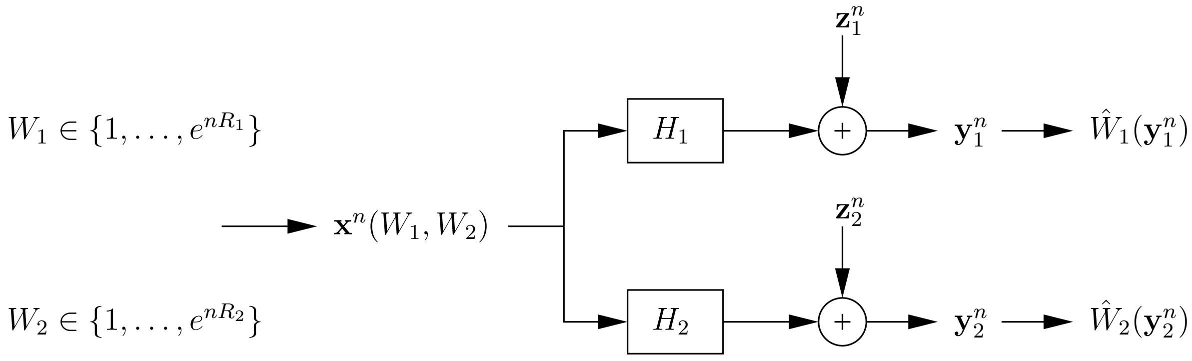

Figure 1: Gaussian MIMO broadcast channel.

where $\mathbf{x}(i) \in \mathbb{R}^t$ is the vector of transmitted symbols and $\mathbf{y}_k(i) \in \mathbb{R}^r$ is the vector of received symbols. The noise vectors $\mathbf{z}_k(i)$ for $k = 1, \ldots, K$ and $i = 1, 2, \ldots$ are i.i.d. white Gaussian noise with identity covariance matrix, I_r. The matrices $H_k \in \mathbb{R}^{r \times t}$, $k = 1, \ldots, K$, represent the channel gains, where the entry $H_k(i, j)$ denotes the channel gain from transmit antenna j to receive antenna i of user k.

A code with rates $(R_1, \ldots, R_K)$ and block length n, denoted by $C(e^{nR_1}, \ldots, e^{nR_K}, n)$, consists of K message sets $\mathcal{W}_k = \{1, \ldots, e^{nR_k}\}$ that contain the intended messages for user $k = 1, \ldots, K$, an encoding function $\mathbf{x}^n(W_1, \ldots, W_K)$ that maps the messages $(W_1, \ldots, W_K) \in \mathcal{W}_1 \times \cdots \times \mathcal{W}_K$ into the transmitted codewords and K decoding functions $\hat{W}_k(\mathbf{y}_k^n)$ that assign the messages $\hat{W}_k \in \mathcal{W}_k$ to the received codewords $\mathbf{y}_k^n$ for $k = 1, \ldots, K$. Average probability of decoding error, $P_e^{(n)}$, is defined as the probability that either of the transmitted messages W_k is decoded erroneously. For the moment, as for the case of Gaussian channels, a total average power constraint is assumed on the transmitted codewords, i.e., for every codeword,

$$\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}(i)^T \mathbf{x}(i) \leq P.$$

More general constraints on the covariance matrix of the transmitted codewords will be considered later in the paper. The rate-tuple $(R_1, \ldots, R_K)$ is said to be achievable if there exists a sequence of $C(e^{nR_1}, \ldots, e^{nR_K}, n)$ codes such that the average probability of error goes to zero as the block length n goes to infinity. The capacity region is then the convex hull of the union of all achievable rates. The capacity region of the Gaussian MIMO BC under total average transmit power constraint P is referred to by $\mathcal{C}_{\mathrm{BC}}(P)$. A two-user Gaussian MIMO BC is shown in Figure 1.

Unlike the scalar BC with $t = r = 1$, the Gaussian MIMO BC in (1) is not degraded in general, hence, the superposition coding and successive decoding of the scalar case is not applicable to the MIMO channel. In the pioneering works [5] [6], Caire and Shamai used Costa's "writing on dirty paper" result [2] to establish an achievable rate region for the Gaussian MIMO BC, commonly referred to as the "Dirty Paper Coding" (DPC) region. They showed that their proposed scheme achieves the sum-rate capacity of a 2-user Gaussian MIMO BC with 2 transmit antennas and one receive antenna at each receiver, and conjectured that this achievable rate region is the capacity. Independent works presented in [8], [9] and [10] further established the optimality of the DPC scheme for the sum-rate. Progress towards establishing this conjecture in general was made in [11] and [12]. By introducing the Degraded Same Marginal (DSM) outer bound, the proof of the conjecture was reduced to that for a degraded Gaussian MIMO BC. The conjecture was finally settled in [13], where the DPC region was proven to be equal to the capacity region. In this paper, an alternative proof of the aforementioned conjecture is provided. The contribution of this paper is to combine several ideas from previous works to provide a rather more intuitive and much simpler converse proof. While the proof employs some previously used ideas, as will be clear later in the paper, it has several key differences with the recent converse of [13].

The rest of this paper is organized as follows: In Section 2, the DPC region is revisited. Based on duality between Gaussian multiple access and broadcast channels, every boundary point of the DPC region is represented as the solution to a convex optimization problem. The proof of the converse for $K = 2$ users is given in Section 3. The proof is extended to more than two users in Section 3.1. In addition, the optimality of the DPC is proven under any arbitrary compact and convex constraint on the transmit covariance matrix in Section 3.2. Section 4 summarizes the paper.

The following notations and abbreviations will be used throughout the paper. Upper case letters denote matrices and boldface letters denote vectors. The ith element of a vector $\mathbf{a}$ is denoted by a_i. The (i, j) entry of a matrix A is denoted by $A(i, j)$. A^T is the transpose of A and $|A|$ is its determinant. An identity matrix of size $n \times n$ is denoted by I_n. $\mathbb{E}(.)$ and $\mathrm{tr}(.)$ denote the expectation and trace operations, respectively. For a symmetric matrix A, $A \succeq 0$ and $A \succ 0$ mean that A is positive semi-definite and positive definite, respectively. The abbreviations BC, MAC, DPC and EPI are used for broadcast channel, multiple access channel, dirty paper coding region and entropy power inequality, respectively.

2 Dirty Paper Coding Region

Dirty paper coding region is constructed based on a surprising result on the capacity of channels with non-causal transmitter side information. Consider a Gaussian MIMO channel given by,

$$\mathbf{y} = \mathbf{x} + \mathbf{s} + \mathbf{z},$$

where $\mathbf{x} \in \mathbb{R}^t$ is the transmitted signal vector. $\mathbf{s}$ and $\mathbf{z} \in \mathbb{R}^t$ are the zero mean Gaussian interference vector with covariance matrix S_s and the Gaussian noise vector with covariance matrix S_z, respectively, and are independent of each other. Assume the interference sequence $\mathbf{s}^n$ is completely known at the transmitter but unknown to the receiver. Therefore to encode the message W, the encoder can choose the transmitted codeword according to W and the interference sequence $\mathbf{s}^n$ as $\mathbf{x}^n(\mathbf{s}^n, W)$. Also assume that there is an average power constraint on each transmitted codeword. It was shown in [3] [4] that the capacity of this channel is the same as if the interference $\mathbf{s}$ does not exist, i.e.,

$$C = \max_{\mathrm{tr}(S_x) \leq P} \frac{1}{2} \log \frac{|S_x + S_z|}{|S_z|}. \quad (2)$$

In other words, interference can be pre-subtracted at the transmitter without increase in transmit power. This result which is known as "writing on dirty paper", can be considered as a generalization of the Costa's work [2] where similar result was obtained for the capacity of a Gaussian scalar channel with i.i.d. Gaussian interference. By using the idea of subtracting interference at the transmitter instead of the receiver, superposition coding can be used in non-degraded Gaussian MIMO BCs. Caire and Shamai [5] [6] used this "writing on dirty paper" idea to establish an achievable rate region for a 2-user Gaussian MIMO BC with 2 transmit antennas and one receive antenna per user. This achievable rate region was referred to as the DPC region and it has been generalized to Gaussian MIMO BCs with arbitrary number of users and antennas [16].

In the DPC scheme, users' messages are encoded successively and the corresponding codewords are added together to form the transmitted codeword. Figure 2 illustrates the DPC scheme for a 2-user Gaussian MIMO BC. Assume the message for user 1 is encoded first by using Gaussian codewords with covariance matrix $\bar{S}_1 = \mathbb{E}(\mathbf{x}_1\mathbf{x}_1^T)$. Consequently, the codeword of user 1, $\mathbf{x}_1^n(W_1)$, can be viewed as a Gaussian interference for user 2 that is completely known to the encoder. Therefore, by the writing on dirty paper result, the encoder can pre-subtract this interference at the transmitter without increase in transmit power. Moreover, the codewords for user 2, $\mathbf{x}_2^n(\mathbf{x}_1^n, W_2)$, are also Gaussian codewords and are statistically independent from $\mathbf{x}_1^n(W_1)$. Let $\bar{S}_2 = \mathbb{E}(\mathbf{x}_2\mathbf{x}_2^T)$ denote the covariance matrix for these codewords. By completely treating the Gaussian interference from user 2 as noise, receiver 1 can achieve the rate R_1 and in the absence of interference from user 1, receiver 2 can achieve the rate R_2 as given below:

$$\begin{aligned} R_1 &= \frac{1}{2} \log \frac{\left|H_1\bar{S}_1H_1^T + H_1\bar{S}_2H_1^T + I_r\right|}{\left|H_1\bar{S}_2H_1^T + I_r\right|}, \\ R_2 &= \frac{1}{2} \log \left|H_2\bar{S}_2H_2^T + I_r\right|. \end{aligned}$$

Since $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ and the codewords are independent, the transmit covariance matrix $\bar{S} = \mathbb{E}(\mathbf{x}\mathbf{x}^T)$ is given by $\bar{S}_1 + \bar{S}_2$. Hence, by using various code-books with covariance matrices $\bar{S}_1, \bar{S}_2 \succeq 0$ that satisfy the average power constraint, $\mathrm{tr}(\bar{S}_1 + \bar{S}_2) \leq P$, an achievable rate region can be constructed. This region can be expanded further by using the other encoding order. The following lemma summarizes the DPC scheme for a K-user Gaussian MIMO BC.

Lemma 2.1 *Given a permutation π on $\{1, \ldots, K\}$ and a set of positive semi-definite matrices $\bar{S}_k$, $k = 1, \ldots, K$, such that tr $\left(\sum_k \bar{S}_k\right) \leq P$, any rate-tuple in the set $\mathcal{F}(\pi, \{H_k\}, \{\bar{S}_k\})$ given as below is achievable for the Gaussian MIMO BC in (1).*

$$\mathcal{F}(\pi, \{H_k\}, \{\bar{S}_k\}) = \left\{\mathbf{R} \in \mathbb{R}_+^K : R_k \leq \bar{R}_k, \quad k = 1, \ldots, K\right\}, \quad (3)$$

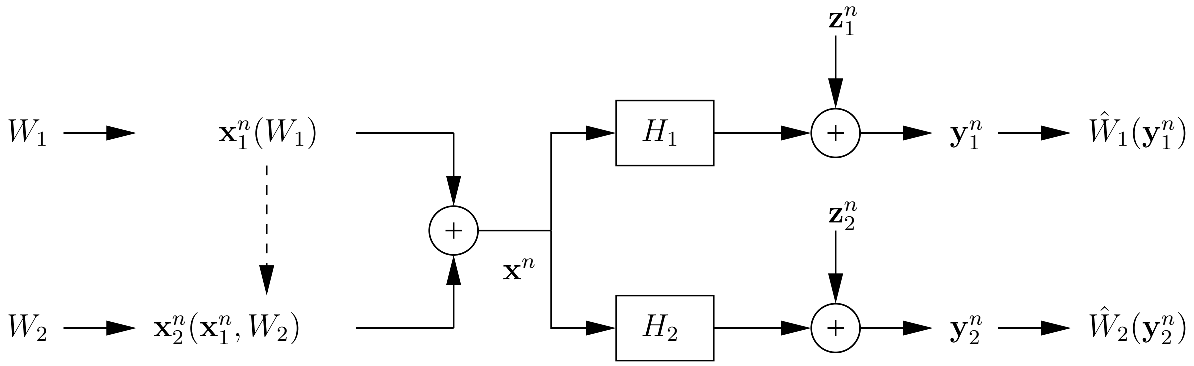

Figure 2: Dirty paper coding for Gaussian MIMO BC.

where for $k = 1, \ldots, K$, $\bar{R}_k$ is defined as,

$$\bar{R}_{\pi(k)} = \frac{1}{2} \log \frac{\left| H_{\pi(k)} \left(\sum_{i=k}^{K} \bar{S}_{\pi(i)} \right) H_{\pi(k)}^T + I_r \right|}{\left| H_{\pi(k)} \left(\sum_{i=k+1}^{K} \bar{S}_{\pi(i)} \right) H_{\pi(k)}^T + I_r \right|}. \tag{4}$$

In this lemma, permutation π determines the encoding order: the message for user $\pi(k)$ is encoded after all messages for preceding users $\pi(i)$, $i < k$, have been encoded. Moreover, $\bar{S}_k$ is the covariance matrix of the transmitted codewords for user k. The proof for K-user case is a straightforward extension of the 2-user case.

The DPC achievable rate region is formally defined in the following.

Definition 2.1 *The DPC achievable rate region of the Gaussian MIMO BC in (1) with average power constraint P, $\mathcal{R}_{\text{DPC}}(P)$, is the convex-hull of the union of all sets $\mathcal{F}(\pi, \{H_k\}, \{\bar{S}_k\})$ over all permutations and admissible covariance matrices, i.e.,*

$$\mathcal{R}_{\text{DPC}}(P) = \boldsymbol{Conv} \left(\bigcup_{\pi, \{\bar{S}_k\}: \bar{S}_k \succeq 0 \ \forall k, \ \sum_k tr(\bar{S}_k) \leq P} \mathcal{F}(\pi, \{H_k\}, \{\bar{S}_k\}) \right). \tag{5}$$

Except for a few special cases, it is almost impossible to characterize the DPC region by examining all possible permutations and admissible transmit covariance matrices. Hence, one might think of using the convexity of this region to characterize its boundary points. However, the rate terms for $\bar{R}_k$ in (4) are not concave functions of $\{\bar{S}_k\}$ and therefore it is very difficult to directly characterize the boundary points of the region $\mathcal{R}_{\text{DPC}}(P)$. In the following section, it is shown how this difficulty can be overcome using the duality theory.

2.1 Alternative Representation of the Dirty Paper Coding Region via Duality

The Gaussian multiple access-broadcast channel duality was observed independently by authors of [7] [8] and [10]. The approach introduced in [7] and [8] is employed for the purpose of this section. There, it was shown that for any permutation π and admissible set of covariance matrices $\{\bar{S}_k\}$, the rates $\bar{R}_k$ in (4) are achievable in a Gaussian multiple access channel that is obtained from the broadcast channel by reversing the roles of the transmitter and the receivers. Specifically, the output of this dual MAC is given by,

$$\mathbf{y} = \sum_{k=1}^{K} H_k^T \mathbf{x}_k + \mathbf{z}, \tag{6}$$

where $\mathbf{y} \in \mathbb{R}^t$ is the received vector, $\mathbf{x}_k \in \mathbb{R}^r$ is the transmitted vector of user k and $\mathbf{z}$ is the receiver's Gaussian noise with covariance matrix I_t. In this dual MAC, the matrices H_k are the same as the ones in (1). Here, $H_k(i,j)$ is the channel coefficient from transmit antenna i of user k to receive antenna j.

The duality result states that for any π and any set of covariance matrices $\{\bar{S}_k\}$, there exists a set of $r \times r$ covariance matrices $\{S_k\}$ such that $\sum_k \operatorname{tr}(S_k) = \sum_k \operatorname{tr}(\bar{S}_k)$ and for $k = 1, \ldots, K$,

$$\frac{1}{2}\log \frac{\left|H_{\pi(k)}\left(\sum_{i\geq k}\bar{S}_{\pi(i)}\right)H_{\pi(k)}^T + I_r\right|}{\left|H_{\pi(k)}\left(\sum_{i>k}\bar{S}_{\pi(i)}\right)H_{\pi(k)}^T + I_r\right|} = \frac{1}{2}\log \frac{\left|\sum_{i\leq k} H_{\pi(i)}^T S_{\pi(i)} H_{\pi(i)} + I_t\right|}{\left|\sum_{i<k} H_{\pi(i)}^T S_{\pi(i)} H_{\pi(i)} + I_t\right|}. \tag{7}$$

Recall that the left-hand side expression is $\bar{R}_{\pi(k)}$ when the covariance matrices of the codewords in the DPC scheme are equal to $\{\bar{S}_k\}$. Moreover, a close look at the righthand side expression reveals that it is equal to $I(\mathbf{x}_{\pi(k)}; \mathbf{y}|\mathbf{x}_{\pi(k+1)}, \ldots, \mathbf{x}_{\pi(K)})$ in the dual MAC in (6) when the users exploit Gaussian code-books with covariance matrices $\{S_k\}$. Therefore, the rates $\bar{R}_k$ in the DPC region of the broadcast channel are achieved by successive decoding in the dual MAC and the users' codewords are decoded in the opposite order of π. Conversely, for any given π and any set of covariance matrices $\{S_k\}$ such that $\sum_k \operatorname{tr}(S_k) \leq P$, there exists a set of covariance matrices $\{\bar{S}_k\}$ that satisfies the equalities in (7) with $\sum_k \operatorname{tr}(S_k) = \sum_k \operatorname{tr}(\bar{S}_k)$. Given that for a Gaussian multiple access channel, Gaussian code-books are optimal, it can be concluded that the DPC region in (5) is equal to the capacity region of the dual MAC under sum power constraint P. Let this region be denoted by $\mathcal{C}_{\text{MAC}}^{sum}(P)$. In short, by duality

$$\mathcal{R}_{\text{DPC}}(P) = \mathcal{C}_{\text{MAC}}^{sum}(P).$$

In order to describe $\mathcal{C}_{\text{MAC}}^{sum}(P)$, recall that by using a Gaussian code-book with covariance matrix S_k for user k, the following set of rates is achievable in the dual multiple access channel [17] [15]:

$$\mathcal{G}(\{H_k^T\}, \{S_k\}) = \left\{\mathbf{R} \in \mathbb{R}_+^K : \sum_{k\in J} R_k \leq \frac{1}{2}\log\left|\sum_{k\in J} H_k^T S_k H_k + I_t\right|, \ \forall\ J \subseteq \{1, \ldots, K\}\right\}. \tag{8}$$

In effect, the capacity region of the dual MAC in (6) under sum power constraint P can be expressed by,

$$\mathcal{C}^{sum}_{\mathrm{MAC}}(P) = \bigcup_{\{S_k\}: S_k \succeq 0 \ \forall k, \sum_k \mathrm{tr}(S_k) \leq P} \mathcal{G}(\{H_k^T\}, \{S_k\}). \tag{9}$$

It is not hard to check that this region is closed. Moreover, the following proposition states that it is also convex without convexification.

Proposition 2.1 *The capacity region of the dual MAC under sum power constraint, as given in (9), is convex.*

The proof of this proposition is given in Appendix A. In addition to the closeness and convexity properties of the set $\mathcal{C}^{sum}_{\mathrm{MAC}}(P)$, according to the following lemma, each of the constituting sets $\mathcal{G}(\{H_k^T\}, \{S_k\})$ has a particular feature that becomes very handy in characterizing the boundary points of the set $\mathcal{C}^{sum}_{\mathrm{MAC}}(P)$.

Lemma 2.2 *For a fixed set of covariance matrices $\{S_k\}$ and any $\mu_1, \ldots, \mu_K \geq 0$, the solution to the optimization problem,*

$$\text{Maximize} \ \sum_{k=1}^{K} \mu_k R_k \qquad \text{Subject to} \ \ \mathbf{R} \in \mathcal{G}(\{H_k^T\}, \{S_k\}),$$

is attained by a permutation π over $\{1, 2, \ldots, K\}$ and a Vertex $\mathbf{R}^\pi$ defined as,

$$R^\pi_{\pi(k)} = \frac{1}{2} \log \frac{\left|\sum_{i=1}^{k} H^T_{\pi(i)} S_{\pi(i)} H_{\pi(i)} + I_t\right|}{\left|\sum_{i=1}^{k-1} H^T_{\pi(i)} S_{\pi(i)} H_{\pi(i)} + I_t\right|} \qquad k = 1, \ldots, K,$$

where π is such that $\mu_{\pi(1)} \geq \mu_{\pi(2)} \geq \cdots \geq \mu_{\pi(K)}$.

The proof of this lemma is based on the *polymatroid* structure of the set $\mathcal{G}(\{H_k^T\}, \{S_k\})$ for a fixed set of $\{S_k\}$ and is provided in [15] and references therein. Note that $\mathbf{R}^\pi$ is achievable by successive decoding scheme in the dual MAC with decoding order determined by the permutation π. The message of user $\pi(K)$ is decoded first and the message for user $\pi(1)$ is decoded last.

All these mentioned properties together with the fact that each set $\mathcal{G}(\{H_k^T\}, \{S_k\})$ is expressed by concave functions of $\{S_k\}$, make the dual representation of the DPC region an easier set to describe. Therefore, in the following, the DPC region is characterized via finding the boundary points of the set $\mathcal{C}^{sum}_{\mathrm{MAC}}(P)$. Since the set $\mathcal{C}^{sum}_{\mathrm{MAC}}(P)$ is a closed and convex set in $\mathbb{R}^K_+$, any point on its boundary (also known as *Pareto* optimal point) can be found by maximizing a weighted sum of the rates [18]. More specifically, any boundary point is a solution to the following optimization problem for some weights $\mu_1, \ldots, \mu_K \geq 0$:

$$\begin{aligned} \text{Maximize} \quad & \sum_{k=1}^{K} \mu_k R_k \\ \text{Subject to} \quad & \mathbf{R} \in \mathcal{C}_{\text{MAC}}^{sum}(P). \end{aligned} \tag{10}$$

Conversely, all the solutions corresponding to all possible selections of the weights constitute the boundary. Without loss of generality, assume all the weights are positive. If $\mu_k = 0$ for some k, the resulting solution can be viewed as a boundary point of the DPC region obtained by positive weights in a broadcast channel derived from the original channel after removing the users with $\mu_k = 0$. Therefore, assume $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_K$. In General, two possible selections for the weights are feasible: The case where not any pair of the weights are equal, e.g., $0 < \mu_1 < \mu_2 < \cdots < \mu_K$ and the case where some of the weights are equal, e.g., $0 < \mu_1 < \cdots < \mu_l = \mu_{l+1} = \cdots = \mu_{l+m} < \cdots < \mu_K$. Characterizing the boundary points corresponding to these two cases are slightly different, therefore, they are considered separately. Lemma 2.3 simplifies further the optimization problem in (10) to characterize the boundary points corresponding to some given weights $\mu_1, \ldots, \mu_K$.

Lemma 2.3 *The boundary point* $\mathbf{R}^*$ *maximizing* $\sum_{k=1}^{K} \mu_k R_k$ *over* $\mathcal{C}_{\text{MAC}}^{sum}(P)$ *for two possible selections of* $\mu_1, \ldots, \mu_K$ *are characterized in the following:*
(i), $0 < \mu_1 < \mu_2 < \cdots < \mu_K$*: in this case* $\mathbf{R}^*$ *is unique and is given by,*

$$R_k^* = \frac{1}{2} \log \frac{\left| \sum_{j=k}^{K} H_j^T S_j^* H_j + I_t \right|}{\left| \sum_{j=k+1}^{K} H_j^T S_j^* H_j + I_t \right|} \qquad k = 1, \ldots, K, \tag{11}$$

where S_k^* *for* $k = 1, \ldots, K$ *are optimal solutions to the following optimization problem (conventionally* $\mu_0 = 0$*):*

$$\text{Maximize} \quad \sum_{k=1}^{K} (\mu_k - \mu_{k-1}) \frac{1}{2} \log \left| \sum_{j=k}^{K} H_j^T S_j H_j + I_t \right| \tag{12}$$

$$\text{Subject to} \quad \sum_{k=1}^{K} tr(S_k) \leq P, \tag{13}$$

$$S_k \succeq 0 \qquad k = 1, \ldots, K. \tag{14}$$

Furthermore, for any optimal S_k^**,* $k = 1, \ldots, K$*,* $\sum_{k=1}^{K} tr(S_k^*) = P$ *and there exists* $\lambda^* \in \mathbb{R}_+$ *and positive semi-definite matrices* Φ_k^* *such that they jointly satisfy the following Karush-Kuhn-Tucker (KKT) optimality conditions for* $k = 1, \ldots, K$*:*

$$H_k \sum_{j=1}^{k} (\mu_j - \mu_{j-1}) \frac{1}{2} \left(\sum_{i=j}^{K} H_i^T S_i^* H_i + I_t \right)^{-1} H_k^T + \Phi_k^* - \lambda^* I_r = \mathbf{0}, \tag{15}$$

$$tr(\Phi_k^* S_k^*) = 0. \tag{16}$$

(ii), $0 < \mu_1 < \cdots < \mu_l = \mu_{l+1} = \cdots = \mu_{l+m} < \cdots < \mu_K$: in this case, the boundary point $\mathbf{R}^*$ may not be unique. In fact, $\mathbf{R}^*$ can be any point in the convex-hull of the vertices $\mathbf{R}^{\sigma_i}$ as given below for all permutations σ_i, $i = 1, \ldots, (m+1)!$, on the set $\mathcal{I} = \{l, \ldots, l+m\}$:

$$R_k^{\sigma_i} = \frac{1}{2} \log \frac{\left| \sum_{j=k}^{K} H_j^T S_j^* H_j + I_t \right|}{\left| \sum_{j=k+1}^{K} H_j^T S_j^* H_j + I_t \right|} \qquad k \in \{1, \ldots, K\} \setminus \mathcal{I}, \tag{17}$$

$$R_{\sigma_i(k)}^{\sigma_i} = \frac{1}{2} \log \frac{\left| \sum_{j=k}^{l+m} H_{\sigma_i(j)}^T S_{\sigma_i(j)}^* H_{\sigma_i(j)} + \sum_{j=l+m+1}^{K} H_j^T S_j^* H_j + I_t \right|}{\left| \sum_{j=k+1}^{l+m} H_{\sigma_i(j)}^T S_{\sigma_i(j)}^* H_{\sigma_i(j)} + \sum_{j=l+m+1}^{K} H_j^T S_j^* H_j + I_t \right|} \quad k \in \mathcal{I}, \tag{18}$$

where S_k^* for $k = 1, \ldots, K$ are solutions to the same optimization problem (12) and satisfy the same KKT optimality conditions in (15)-(16). Moreover, any such $\mathbf{R}^*$ satisfies the following equalities:

$$R_k^* = \frac{1}{2} \log \frac{\left| \sum_{j=k}^{K} H_j^T S_j^* H_j + I_t \right|}{\left| \sum_{j=k+1}^{K} H_j^T S_j^* H_j + I_t \right|} \qquad k \in \{1, \ldots, K\} \setminus \mathcal{I}, \tag{19}$$

$$\sum_{k=l}^{l+m} R_k^* = \frac{1}{2} \log \frac{\left| \sum_{j=l}^{K} H_j^T S_j^* H_j + I_t \right|}{\left| \sum_{j=l+m+1}^{K} H_j^T S_j^* H_j + I_t \right|}. \tag{20}$$

Proof: First consider the case $0 < \mu_1 < \cdots < \mu_K$. Since the set $\mathcal{C}_{\mathrm{MAC}}^{sum}(P)$ as defined in (9) is closed and convex, each of its boundary points must belong to a set $\mathcal{G}(\{H_k^T\}, \{S_k\})$, for some covariance matrices $\{S_k\}$. Using the result of Lemma 2.2, the vertex given by

$$R_k = \frac{1}{2} \log \frac{\left| \sum_{j=k}^{K} H_j^T S_j H_j + I_t \right|}{\left| \sum_{j=k+1}^{K} H_j^T S_j H_j + I_t \right|} \qquad k = 1, \ldots, K,$$

maximizes $\sum_k \mu_k R_k$ over $\mathcal{G}(\{H_k^T\}, \{S_k\})$. Moreover, this point is achievable by successive decoding in the dual MAC where the message for user k is decoded kth in the order. Substituting the rate terms for R_k in (10) and including the sum power constraint, the optimization problem in (12) is obtained.

It is easy to show that the cost function of this optimization problem is continuous in $\{S_k\}$ for any norm on the space of symmetric matrices. As is shown in [18], this function is actually differentiable with respect to these variables. Moreover, the optimization domain defined by the constraints (13) and (14) is closed and compact for the same norm. Hence, by *Weierstrass* theorem [19], there exist S_k^* for $k = 1, \ldots, K$ that achieve the maximum. In addition, this optimization problem is convex since it has a concave cost function and convex constraints in the covariance matrices $\{S_k\}$ as in (13) and (14). Also the Slater condition holds and the feasible region has an interior point for any $P > 0$. Thus, any optimal solution of (12) must satisfy the Karush-Kuhn-Tucker (KKT) optimality conditions and vice versa [18]. To obtain the KKT conditions, let $\lambda \geq 0$ be the dual variable associated with the sum power constraint in (13) and the matrix $\Phi_k \succeq 0$ be the dual variable associated with the positive semi-definite constraint on S_k given in (14) for $k = 1, \ldots, K$. Then the Lagrangian for this optimization problem can be expressed as,

$$\begin{aligned} \mathcal{L}(\{S_k\}, \{\Phi_k\}, \lambda) &= \sum_{k=1}^{K} (\mu_k - \mu_{k-1}) \frac{1}{2} \log \left| \sum_{j=k}^{K} H_j^T S_j H_j + I_t \right| \\ &\quad + \sum_{k=1}^{K} \mathrm{tr}(S_k \Phi_k) - \lambda (\sum_{k=1}^{K} \mathrm{tr}(S_k) - P). \end{aligned}$$

Taking the derivative of the Lagrangian with respect to S_k for some norm on the space of symmetric matrices, yields the left-hand side expression in (15) for $k = 1, \ldots, K$ [18]. The conditions in (16) are known as the complementary slackness conditions. Since the optimum value is achieved, there exist feasible S_k^*, Φ_k^* for $k = 1, \ldots, K$ and λ^* that satisfy the KKT conditions in (15)-(16). S_k^* are referred to as the primal optimal solutions while Φ_k^* and λ^* are referred to as the dual optimal solutions. Note that since for any k, $\log \left| \sum_{j=k}^{K} H_j^T \alpha S_j H_j + I_t \right|$ is a strictly increasing function of $\alpha \in \mathbb{R}_+$, the power constraint must hold with equality, i.e., $\sum_k \mathrm{tr}(S_k^*) = P$. Otherwise, all S_k^*s can be scaled up by a factor $\alpha > 1$ to increase the cost function while still satisfying the trace constraint. Also all the matrix terms on the left-hand sides of (15) are positive semi-definite and the matrix terms involving S_k^*s cannot be equal to all zero matrix. Therefore, the optimal λ^* must be positive. For this choice of the weights, it can be shown that the boundary point $\mathbf{R}^*$ is also unique[1].

Next consider the case $0 < \mu_1 < \cdots < \mu_l = \mu_{l+1} = \cdots = \mu_{l+m} < \cdots < \mu_K$. As in the case of unequal weights, each boundary point must belong to a set $\mathcal{G}(\{H_k^T\}, \{S_k\})$ for some feasible covariance matrices $\{S_k\}$. According to Lemma 2.2, $\sum_k \mu_k R_k$ is maximized over $\mathcal{G}(\{H_k^T\}, \{S_k\})$ by all the vertices that have the following decoding orders: user k for $k \notin \{l, \ldots, l+m\}$ is decoded kth in the order. However, since $\mu_l = \cdots = \mu_{l+m}$, this lemma does not specify any decoding order for the users in $\{l, \ldots, l+m\}$. Therefore, by choosing various decoding orders specified by the permutations σ_i for $i = 1, \ldots, (m+1)!$ on these $m+1$ users, all the vertices $\mathbf{R}^{\sigma_i}$ that maximize $\sum_k \mu_k R_k$ can be found as given in (17)-(18). Recall that there is no other vertex that maximizes $\sum_k \mu_k R_k$. Referring to the polymatroid structure of the set $\mathcal{G}(\{H_k^T\}, \{S_k\})$ for a fixed set of $\{S_k\}$ [15], the convex-hull of these $(m+1)!$ vertices constitutes a m-dimensional boundary surface of $\mathcal{G}(\{H_k^T\}, \{S_k\})$ and clearly any point on this convex-hull maximizes $\sum_k \mu_k R_k$. Note that some or all of these points may coincide resulting in a smaller dimensional surface (possibly a single point). However, in general, they produce a m-dimensional surface. It is easy to verify that for all

[1] Uniqueness of this point is a direct consequence of strict concavity of $\log |.|$ function, however, it is of the least significance to the proof and is included here for the sack of completeness.

the vertices $\mathbf{R}^{\sigma_i}$ for $i = 1, \dots, (m+1)!$,

$$
\begin{aligned}
R_k^{\sigma_i} &= \frac{1}{2} \log \frac{\left| \sum_{j=k}^{K} H_j^T S_j H_j + I_t \right|}{\left| \sum_{j=k+1}^{K} H_j^T S_j H_j + I_t \right|} \qquad k \in \{1, \dots, K\} \setminus \mathcal{I}, \\
\sum_{k=l}^{l+m} R_k^{\sigma_i} &= \frac{1}{2} \log \frac{\left| \sum_{j=l}^{K} H_j^T S_j H_j + I_t \right|}{\left| \sum_{j=l+m+1}^{K} H_j^T S_j H_j + I_t \right|}.
\end{aligned}
$$

Since $\mu_l = \cdots = \mu_{l+m}$, after substituting the rate terms for these vertices in $\sum_k \mu_k R_k$ and taking into account the second equality above, the same optimization problem in (12) is obtained. The solution to this problem may not be unique, however, any set of optimal $\{S_k^*\}$ satisfies the KKT conditions in (15)-(16) and identifies the vertices $\mathbf{R}^{\sigma_i}$. The boundary point $\mathbf{R}^*$ may be chosen as any point in the convex-hull of these vertices and clearly satisfies the equalities in (19) and (20). Note that the aforementioned boundary point $\mathbf{R}^*$ may no longer be achievable by the successive decoding scheme in the dual MAC. Hence, it may not be achievable by only using the DPC scheme in the broadcast channel. Nevertheless, the DPC scheme achieves each of the vertices $\mathbf{R}^{\sigma_i}$ and by time sharing among the codes achieving these vertices, $\mathbf{R}^*$ can be achieved. □

This result will be exploited in the next section to prove the optimality of the DPC region.

Figure 3 sketches the DPC region and shows a boundary point (R_1^*, R_2^*) for some μ_1, μ_2.

3 Optimality of the Dirty Paper Coding Scheme

Theorem 3.1 $\mathcal{R}_{\text{DPC}}(P)$ *is the capacity region,* $\mathcal{C}_{\text{BC}}(P)$.

To prove this theorem, the same approach proposed in [11], [12] and [13] is used: the boundary of the DPC region is partitioned into several segments (possibly single points) and the converse is proven independently for each segment. The main idea of this approach is to exploit the known results on the capacity of degraded broadcast channels. By partitioning the boundary of the DPC region into several segments, a degraded broadcast channel is constructed for each boundary segment, $\mathcal{B}$, with the following properties: First, it has the same segment $\mathcal{B}$ on the boundary of its DPC region. In other words, for that particular segment, the DPC scheme performs at best the same in the degraded channel and the original channel. Second, the capacity region of the degraded channel contains the capacity region of the original channel. Then, by using the known results on the capacity of degraded broadcast channels, it is shown that the DPC scheme is optimal for the degraded channel, hence, this channel has the segment $\mathcal{B}$ on the boundary of its capacity region. Since the capacity region of the degraded channel contains the capacity region of the original channel, it is concluded that $\mathcal{B}$ is also on the boundary of the capacity region of the original channel. The same

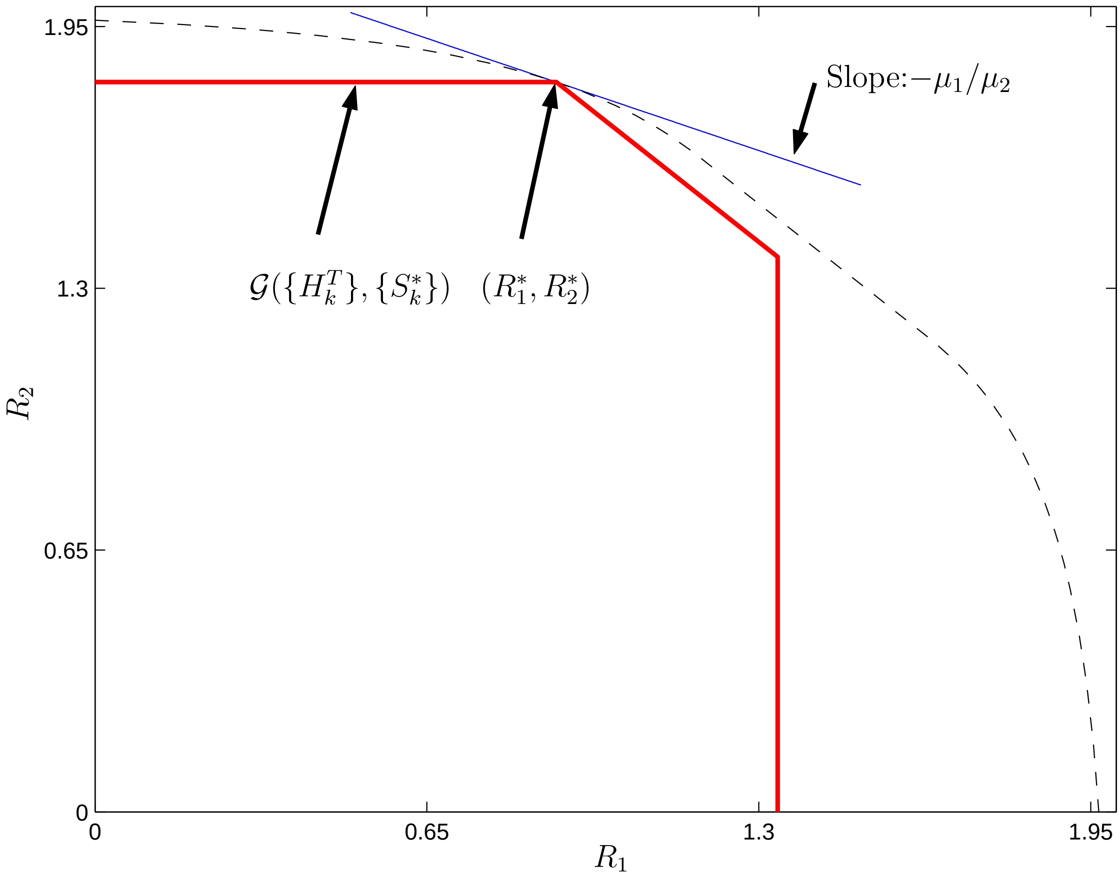

Figure 3: Characterizing the boundary points of the DPC region via duality.

argument is used for all the boundary segments to prove the optimality of the DPC scheme for the whole region.

The authors of [11] and [12] successfully constructed these degraded broadcast channels for each boundary segment, however, they failed to prove the optimality of the DPC scheme for these degraded broadcast channels. Obviously, the choice of these degraded channels has a significant effect on the simplicity of the converse proof.

In this section, the converse proof is given for each boundary point of $\mathcal{R}_{\text{DPC}}(P)$. A degraded broadcast channel for the boundary point $\mathbf{R}^*$ that maximizes $\sum_{k=1}^{K} \mu_k R_k$, for a given $\mu_1, \ldots, \mu_K \geq 0$, is defined based on the optimality conditions given in Lemma 2.3. In Lemma 3.1, it is shown that $\mathbf{R}^*$ is on the boundary of the DPC region of this degraded channel (first property). Furthermore, it is proven that this degraded channel has a larger capacity region compared to the original channel (second property). Using the entropy power inequality, it is then proven that $\mathbf{R}^*$ also lies on the boundary of the capacity region of the degraded broadcast channel (Lemma 3.2).

While the proof presented in this section borrows its main idea from the previous works including the recent converse by Weingarten, Steinberg and Shamai (WSS) [13], it has several key differences. The converse by WSS is initially proven for a particular class of degraded MIMO broadcast channels referred to as Aligned Degraded Broadcast Channels (ADBC), while the proof proposed here directly applies to general Gaussian MIMO broadcast channels. Moreover, in [13], the converse for the ADBC channels is proven through the definition of the enhanced ADBC channel with certain properties that make it possible to employ the entropy power inequality. Although, this enhanced ADBC channel shows up in the proposed proof here, the way this channel is obtained is totally different from the approach of WSS. In this section, this channel is initially defined based on the optimality conditions for a convex optimization problem and it will turn out that it has all the properties of the enhanced channel. While in [13], the existence of such a channel is proven mainly using non-convex optimization techniques. As the last step of the WSS proof, the converse is generalized to the larger class of Aligned Multiple-input-multiple-output Broadcast Channels (AMBC). Using the result for the AMBC channels, the proof is then extended to the general Gaussian MIMO BC by showing that the capacity region of a MIMO BC can be expressed as the limit of the capacity regions of a sequence of AMBC channels as some of the eigenvalues of the noise covariance matrices go to infinity. Nevertheless, this limiting argument is not needed in the converse proof of this section.

To focus mainly on the key steps of the proof instead of the details and simplify the presentation, first, the optimality of the DPC scheme is proven for $K = 2$ users. Study of the general $K > 2$ user case is postponed to Section 3.1.

Now the details of the proof are given. Consider the boundary point $\mathbf{R}^*$ of $\mathcal{R}_{\text{DPC}}(P)$ corresponding to given $0 < \mu_1 < \mu_2$ as characterized in Lemma 2.3. Recall that for $K = 2$, the choices of $\mu_1 = 0$ or $\mu_2 = 0$ and $\mu_1 = \mu_2$ correspond to the capacities of the individual users and the maximum sum-rate point of the DPC region, respectively. These points are of no interest since optimality of the DPC scheme is already known for them (see [8], [9] and [10]). Thus, it only remains to consider the cases $0 < \mu_1 < \mu_2$ or $0 < \mu_2 < \mu_1$. Since the proof for the latter case is identical to the proof for the former case, it is sufficient to only consider $0 < \mu_1 < \mu_2$. For these given weights, define the $t \times t$ symmetric matrices,

$$Q_1 = \frac{\mu_1}{2\lambda^*}\left(H_1^T S_1^* H_1 + H_2^T S_2^* H_2 + I_t\right)^{-1}, \tag{21}$$

$$Q_2 = \frac{\mu_1}{2\lambda^*}\left(H_1^T S_1^* H_1 + H_2^T S_2^* H_2 + I_t\right)^{-1} + \frac{\mu_2 - \mu_1}{2\lambda^*}\left(H_2^T S_2^* H_2 + I_t\right)^{-1}, \tag{22}$$

where S_1^*, S_2^* and λ^* are, respectively, the primal and dual optimal solutions of the optimization problem (12) in Lemma 2.3. Clearly, these matrices are positive definite. Note that both Q_1 and Q_2 depend on the *weight vector* $\boldsymbol{\mu} = (\mu_1, \mu_2)$. This dependency is not explicitly included for notational simplicity. In the following, a degraded MIMO BC is defined corresponding to the boundary point of $\mathcal{R}_{\text{DPC}}(P)$ under consideration.

Definition 3.1 *For a given weight vector* $\boldsymbol{\mu}$ *and its corresponding boundary point* $\mathbf{R}^*$ *of* $\mathcal{R}_{\text{DPC}}(P)$*, define the DBC(*$\boldsymbol{\mu}$*) channel as,*

$$\mathbf{y}_k = \mathbf{x} + \mathbf{z}_k \quad k = 1, 2, \tag{23}$$

where $\mathbf{x}, \mathbf{y}_1$ *and* $\mathbf{y}_2 \in \mathbb{R}^t$ *are the channel input and output vectors, respectively, and* $\mathbf{z}_1, \mathbf{z}_2$ *are Gaussian noise vectors with covariance matrices* Q_1 *and* Q_2*, respectively. Further assume the same total average transmit power* P *for this channel.*

It is immediate from definition of Q_1 and Q_2 that $0 \prec Q_1 \prec Q_2$. This choice of Q_1 and Q_2 ensures that DBC($\boldsymbol{\mu}$) is statistically degraded, a property that will be used later to establish its capacity region. The following lemma shows that the boundary of the DPC region of DBC($\boldsymbol{\mu}$) is tangent to the boundary of $\mathcal{R}_{\text{DPC}}(P)$ at $\mathbf{R}^*$, hence, this point is achievable by the DPC scheme in this channel.

Lemma 3.1 *The point* $\mathbf{R}^*$ *maximizes* $\mu_1 R_1 + \mu_2 R_2$ *over the DPC region of DBC(*$\boldsymbol{\mu}$*) denoted by* $\mathcal{R}_{\text{DPC}}^{\text{DBC}(\boldsymbol{\mu})}(P)$.

The proof of this lemma is given in Appendix B. Figure 4 shows the DPC regions for both the original channel and the degraded channel, DBC($\boldsymbol{\mu}$), defined for the point $\mathbf{R}^*$.

Before proceeding to prove that the point $\mathbf{R}^*$ also lies on the boundary of the capacity region of DBC($\boldsymbol{\mu}$), consider the transmit covariance matrices for the DBC($\boldsymbol{\mu}$) channel that achieve $\mathbf{R}^*$ by the DPC scheme. Denote these matrices by $\bar{\Gamma}_1^*$ and $\bar{\Gamma}_2^*$. Lemma 3.3 of Section 3.1 shows that these matrices are given by,

$$\begin{aligned}\bar{\Gamma}_1^* &= \frac{\mu_1}{2\lambda^*}\left(H_2^T S_2^* H_2 + I_t\right)^{-1} - Q_1 \\ &= \frac{\mu_1}{2\lambda^*}\left((H_2^T S_2^* H_2 + I_t)^{-1} - (H_1^T S_1^* H_1 + H_2^T S_2^* H_2 + I_t)^{-1}\right),\end{aligned} \tag{24}$$

$$\begin{aligned}\bar{\Gamma}_2^* &= \frac{\mu_2}{2\lambda^*} I_t - \bar{\Gamma}_1^* - Q_2 \\ &= \frac{\mu_2}{2\lambda^*}\left(I_t - (H_2^T S_2^* H_2 + I_t)^{-1}\right),\end{aligned} \tag{25}$$

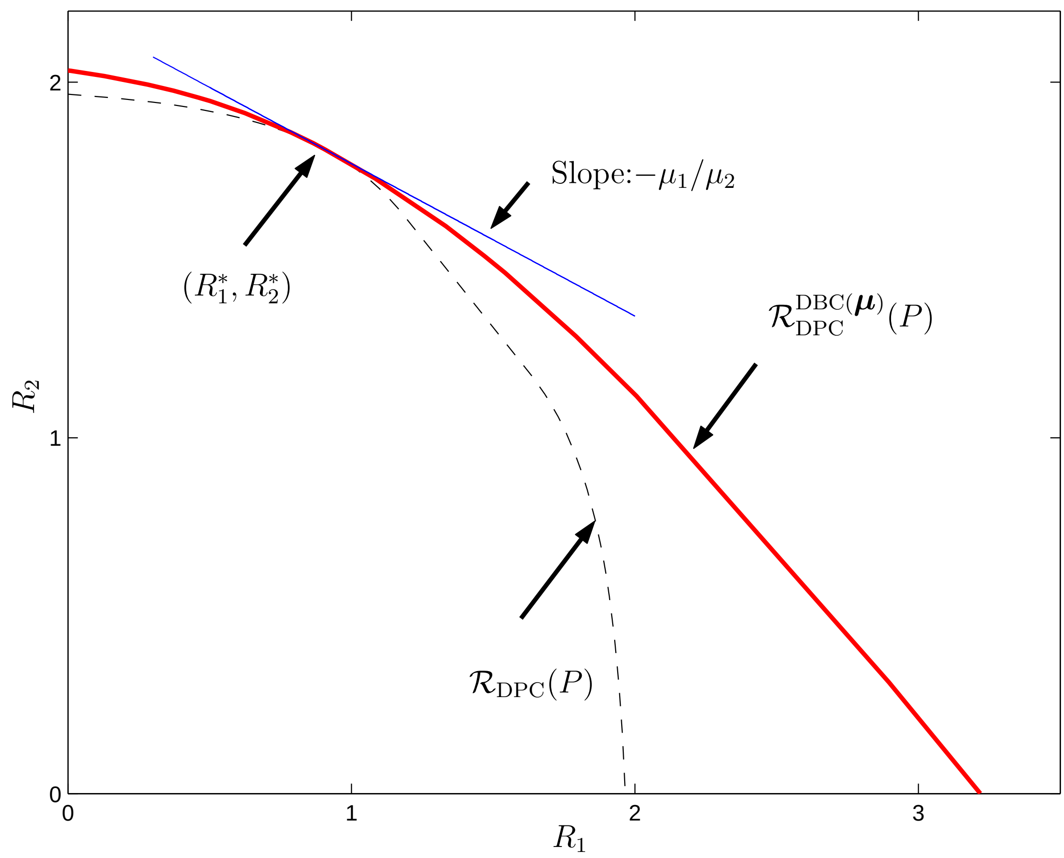

Figure 4: The point (R_1^*, R_2^*) lies on the boundaries of the DPC regions for the original broadcast channel and the degraded channel, DBC($\boldsymbol{\mu}$).

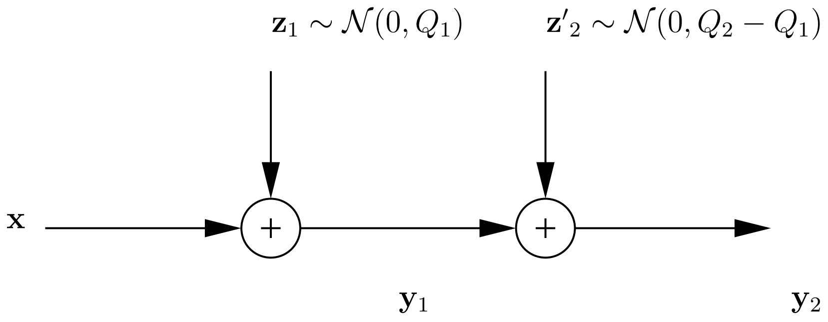

Figure 5: Physically degraded broadcast channel with the same capacity region as DBC($\boldsymbol{\mu}$).

where S_1^*, S_2^* and λ^* are the optimal solutions to the optimization problem in (12). Moreover, this lemma expresses R_1^*, R_2^* in terms of $\bar{\Gamma}_1^*$ and $\bar{\Gamma}_2^*$ as,

$$\begin{aligned} R_1^* &= \frac{1}{2}\log\frac{\left|\bar{\Gamma}_1^* + Q_1\right|}{|Q_1|}, \\ R_2^* &= \frac{1}{2}\log\frac{\left|\bar{\Gamma}_1^* + \bar{\Gamma}_2^* + Q_2\right|}{\left|\bar{\Gamma}_1^* + Q_2\right|}. \end{aligned}$$

Lemma 3.2 *The point $\mathbf{R}^*$, which was shown to be on the boundary of the DPC region of DBC($\boldsymbol{\mu}$) is also on the boundary of its capacity region.*

Proof: The method of proof by contradiction is employed to verify that (R_1^*, R_2^*) is on the boundary of the capacity region of DBC($\boldsymbol{\mu}$). The steps of the proof is very similar to the Bergmans' convers given for the scalar case [1]. Since $Q_1 \prec Q_2$, DBC($\boldsymbol{\mu}$) has the same marginal transition probability distributions, and therefore, the same capacity region as a physically degraded broadcast channel given by,

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x} + \mathbf{z}_1, \\ \mathbf{y}_2 &= \mathbf{y}_1 + \mathbf{z'}_2, \end{aligned}$$

where $\mathbf{z}_1$ and $\mathbf{z'}_2$ are independent Gaussian noises with covariance matrices equal to Q_1 and $Q_2 - Q_1$, respectively (see Figure 5). To be able to use the entropy power inequality, this degraded version of DBC($\boldsymbol{\mu}$) is used for the capacity analysis. First assume (R_1^*, R_2^*) is not on the boundary and lies within the capacity region of DBC($\boldsymbol{\mu}$). Therefore, there exists a rate-pair (R_1, R_2) in the capacity region and an arbitrary small $\delta > 0$ such that $R_k^* + 2\delta \le R_k$ for $k = 1, 2$. Consider $C(e^{nR_1}, e^{nR_2}, n)$, an arbitrary sequence of codes each with block length n and rates (R_1, R_2) such that the average probability of decoding error, $P_e^{(n)}$, vanishes as $n \to \infty$. Let $\mathbf{x}^n$ denote the nt by 1 stacked vector of the transmitted symbols, $\mathbf{x}^n = [\mathbf{x}(1)^T \cdots \mathbf{x}(n)^T]^T$ and define the noise vectors $\mathbf{z}_1^n$, $\mathbf{z}_2'^n$ and corresponding output vectors $\mathbf{y}_1^n$, $\mathbf{y}_2^n$ similarly. By Fano's inequality, for the codes under consideration with large enough block-length, n, the following inequalities hold:

$$R_1^* + \delta \quad \leq \quad R_1 - \delta \leq \frac{1}{n} I(W_1; \mathbf{y}_1^n | W_2), \tag{26}$$

$$R_2^* + \delta \quad \leq \quad R_2 - \delta \leq \frac{1}{n} I(W_2; \mathbf{y}_2^n), \tag{27}$$

where W_1, W_2 are intended messages for user 1 and user 2, respectively. By expanding the mutual information term, (26) is reduced to,

$$R_1^* + \delta \leq \frac{1}{n} h(\mathbf{y}_1^n | W_2) - \frac{1}{n} h(\mathbf{y}_1^n | W_1, W_2),$$

where $h(.)$ is the differential entropy function. Note that $\mathbf{y}_1^n$, $\mathbf{y}_2^n$ are obtained by addition of two independent Gaussian noise vectors to $\mathbf{x}^n$, hence, they both have densities. $h(\mathbf{y}_1^n | W_1, W_2) = h(\mathbf{z}_1^n) = \frac{n}{2} \log(2\pi e)^t |Q_1|$ and $R_1^* = \frac{1}{2} \log \left| \bar{\Gamma}_1^* + Q_1 \right| - \frac{1}{2} \log |Q_1|$, from these equalities the following lower bound on $h(\mathbf{y}_1^n | W_2)$ can be established:

$$\frac{1}{n} h(\mathbf{y}_1^n | W_2) \geq \frac{1}{2} \log(2\pi e)^t \left| \bar{\Gamma}_1^* + Q_1 \right| + \delta. \tag{28}$$

Now since $\mathbf{z}'^n_2$ is independent of $(W_1, W_2, \mathbf{z}_1^n)$, and conditioned on W_2, $\mathbf{y}_2^n = \mathbf{y}_1^n + \mathbf{z}'^n_2$ and $\mathbf{y}_1^n$ have densities, the entropy power inequality [17] can be applied to obtain,

$$\exp\left(\frac{2}{nt} h(\mathbf{y}_2^n | W_2)\right) \geq \exp\left(\frac{2}{nt} h(\mathbf{y}_1^n | W_2)\right) + \exp\left(\frac{2}{nt} h(\mathbf{z}'^n_2)\right). \tag{29}$$

Employing the lower bound obtained in (28) in (29) and substituting $h(\mathbf{z}'_2) = \frac{1}{2} \log(2\pi e)^t |Q_2 - Q_1|$, the inequality (29) is reduced to,

$$\exp\left(\frac{2}{nt} h(\mathbf{y}_2^n | W_2)\right) \geq 2\pi e \left(\left| \bar{\Gamma}_1^* + Q_1 \right|^{\frac{1}{t}} + |Q_2 - Q_1|^{\frac{1}{t}} \right) + \delta',$$

for some small $\delta' > 0$. However, as illustrated in the following, the expressions for $\bar{\Gamma}_1^*$, Q_1 and Q_2 in (24), (21) and (22) reveal that the two matrix expressions on the right-hand side, $(\bar{\Gamma}_1^* + Q_1)$ and $(Q_2 - Q_1)$, are scaled versions of each other:

$$\bar{\Gamma}_1^* + Q_1 \quad = \quad \frac{\mu_1}{2\lambda^*} (H_2^T S_2^* H_2 + I_t)^{-1},$$

$$Q_2 - Q_1 \quad = \quad \frac{(\mu_2 - \mu_1)}{2\lambda^*} (H_2^T S_2^* H_2 + I_t)^{-1}.$$

In effect, they satisfy the following equality:

$$\left| \bar{\Gamma}_1^* + Q_1 \right|^{\frac{1}{t}} + |Q_2 - Q_1|^{\frac{1}{t}} = \left| \bar{\Gamma}_1^* + Q_2 \right|^{\frac{1}{t}}.$$

This equality yields a lower bound on $h(\mathbf{y}_2^n | W_2)$ as given below:

$$\frac{1}{n} h(\mathbf{y}_2^n | W_2) \geq \frac{1}{2} \log(2\pi e)^t \left| \bar{\Gamma}_1^* + Q_2 \right| + \delta'', \tag{30}$$

where $\delta'' > 0$ is some small positive constant. The lower bound (30) can be used in the Fano's inequality (27) to obtain a lower bound on $h(\mathbf{y}_2^n)$ as

$$\begin{aligned} \frac{1}{n} h(\mathbf{y}_2^n) &\geq R_2^* + \frac{1}{2}\log(2\pi e)^t \left|\bar{\Gamma}_1^* + Q_2\right| + \delta + \delta'' \\ &\geq \frac{1}{2}\log(2\pi e)^t \left|\bar{\Gamma}_1^* + \bar{\Gamma}_2^* + Q_2\right| + \delta + \delta'', \end{aligned}$$

where the second term is obtained by substituting $R_2^* = \frac{1}{2}\log\left|\bar{\Gamma}_1^* + \bar{\Gamma}_2^* + Q_2\right| - \frac{1}{2}\log\left|\bar{\Gamma}_1^* + Q_2\right|$. However, $\mathbf{y}_2^n = \mathbf{x}^n + \mathbf{z}_1^n + \mathbf{z}'^n_2$ is the transmitted codeword corrupted by an additive Gaussian noise that is i.i.d. over transmissions and on each transmission it has covariance matrix Q_2. In other words, $\mathbf{z}_1^n + \mathbf{z}'^n_2$ has nt by nt block-diagonal covariance matrix with Q_2 on each diagonal block:

$$\begin{bmatrix} Q_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & Q_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & Q_2 \end{bmatrix}.$$

Moreover, there is an average power constraint on $\mathbf{x}^n$,

$$\sum_{i=1}^{n} \mathbb{E}(\mathbf{x}(i)^T \mathbf{x}(i)) = \mathbb{E}\left(\mathbf{x}^{nT}\mathbf{x}^n\right) \leq nP.$$

Therefore, by water-filling conditions [17], $h(\mathbf{y}_2^n)$ is maximized for a random vector $\mathbf{x}^n$ that has i.i.d. Gaussian $\mathbf{x}(i)$ for $i = 1, \ldots, n$. In addition, each $\mathbf{x}(i)$ has zero mean and covariance matrix that has the same eigenvectors as the noise covariance matrix Q_2 and its eigenvalues water-fill the eigenvalues of Q_2 . It is not hard to show that the optimal positive semi-definite covariance matrix $\Sigma = \mathbb{E}(\mathbf{x}(i)\mathbf{x}(i)^T)$ that maximizes $h(\mathbf{y}_2^n)$ subject to the power constraint must satisfy the water-filling conditions as given below,

$$\begin{aligned} (\Sigma + Q_2)^{-1} + \Theta &= \alpha I_t, \\ \mathrm{tr}(\Sigma) &= P, \\ \mathrm{tr}(\Sigma\Theta) &= 0, \end{aligned}$$

where Θ is a $t \times t$ positive semi-definite matrix and α is a positive real number. Form (25), it can be seen that $\bar{\Gamma}_1^* + \bar{\Gamma}_2^* + Q_2 = \frac{\mu_2}{2\lambda^*} I_t$ and furthermore, in the proof of Lemma 3.3, it is shown that $\mathrm{tr}(\bar{\Gamma}_1^* + \bar{\Gamma}_2^*) = P$. Therefore, the transmit covariance matrix $\bar{\Gamma}_1^* + \bar{\Gamma}_2^*$ satisfies the water-filling conditions for $\Theta = \mathbf{0}$ and $\alpha = 2\lambda^*/\mu_2$ and maximizes $h(\mathbf{y}_2^n)$. In effect, $\frac{1}{n}h(\mathbf{y}_2^n) \leq \frac{1}{2}\log(2\pi e)^t \left|\bar{\Gamma}_1^* + \bar{\Gamma}_2^* + Q_2\right|$, that contradicts the previous inequality. Therefore (R_1^*, R_2^*) lies on the boundary of the capacity region of DBC($\boldsymbol{\mu}$). □

To complete the proof and show that $\mathbf{R}^*$ is on the boundary of $\mathcal{C}_{\mathrm{BC}}(P)$, it remains to show that the capacity region of DBC($\boldsymbol{\mu}$) contains the capacity region of the original broadcast channel. Since $\mathbf{R}^*$ is on the boundary of the capacity region of DBC($\boldsymbol{\mu}$) that contains $\mathcal{C}_{\mathrm{BC}}(P)$, it must be on the boundary of $\mathcal{C}_{\mathrm{BC}}(P)$ as well.

Let (R_1, R_2) be a rate-pair in $\mathcal{C}_{\mathrm{BC}}(P)$. To prove $\mathcal{C}_{\mathrm{BC}}(P)$ is contained in the capacity region of DBC($\boldsymbol{\mu}$), it is sufficient to show that any code achieving the rate-pair (R_1, R_2) in the original broadcast channel with arbitrary small probability of decoding error, can be used in DBC($\boldsymbol{\mu}$) to achieve the same rates with at least the same probability of decoding error. Consider a code for the original broadcast channel with rates (R_1, R_2) and arbitrary small probability of error and assume this code is used in DBC($\boldsymbol{\mu}$). To be able to decode each codeword, appropriate decoder is constructed for receiver k of DBC($\boldsymbol{\mu}$) as described in the following:

1. Receiver k multiplies each received symbol $\mathbf{y}_k(i)$ by H_k.
2. Adds an i.i.d Gaussian noise vector with covariance matrix $I_r - H_k Q_k H_k^T$ to the resulting symbols for Q_k as given in (21)-(22).
3. The resulting symbols are decoded using the same decoding rule as in the original broadcast channel.

After step 1, receiver k obtains a codeword that is statistically the same as the one received in a Gaussian MIMO BC with channel matrix H_k and noise covariance matrix $H_k Q_k H_k^T$ for receiver k. Note that equalities given in (15) ensures that $H_k^T Q_k H_k \preceq I_r$ for $k = 1, 2$. Therefore, there exists a Gaussian noise vector with covariance matrix as given in step 2. By step 2, the resulting codeword is statistically the same as the one passed through a broadcast channel with channel matrix H_k and noise covariance matrix $H_k Q_k H_k^T + I_r - H_k Q_k H_k^T = I_r$ for receiver k. Thus, the same decoding functions as in the original broadcast channel can be used to decode each message with at least the same probability of error. This completes the converse proof for $K = 2$ users.

3.1 Extension to More than Two Users

In this section, the converse proof is extended to more than two users. Except for some technical details, the extended proof follows exactly the same line of reasoning as for the two user case. Let $\mathbf{R}^*$ be the boundary point of $\mathcal{R}_{\text{DPC}}(P)$ corresponding to some given weights $\mu_1, \ldots, \mu_k \geq 0$. For the converse proof, it is only sufficient to prove the optimality of the DPC scheme for the boundary points that correspond to positive weights. If $\mu_k = 0$ for some k, then $\mathbf{R}^*$ will essentially lie on the boundary of the DPC region of a $K-1$ user broadcast channel obtained by removing user k from the original channel. Hence, by induction on K and assuming that the converse holds for $K-1$ users, this point will be on the boundary of the capacity region of this $K-1$ user broadcast channel, which is in fact the boundary segment of the capacity region of the K user original channel corresponding to $R_k = 0$. Consequently, $\mathbf{R}^*$ will lie on the boundary of the capacity region of the K user broadcast channel. Therefore, without loss of generality assume $0 < \mu_1 \leq \cdots \leq \mu_K$ and consider the corresponding boundary point $\mathbf{R}^*$ as specified in Lemma 2.3.

Parallel to Section 3, for the boundary point $\mathbf{R}^*$ define the matrices $Q_k \in \mathbb{S}^t$ as,

$$Q_k = \frac{1}{2\lambda^*} \sum_{j=1}^{k} (\mu_j - \mu_{j-1}) \left(\sum_{i=j}^{K} H_i^T S_i^* H_i + I_t \right)^{-1} \qquad k = 1, \ldots, K. \tag{31}$$

Also let the degraded broadcast channel DBC($\boldsymbol{\mu}$) be defined as before:

$$\mathbf{y}_k = \mathbf{x} + \mathbf{z}_k, \qquad \text{for } k = 1, \ldots, K,$$

where $\mathbf{z}_k$ is a white Gaussian noise vector with covariance matrix equal to Q_k. Note that $0 \prec Q_1 \preceq Q_2 \preceq \cdots \preceq Q_K$ and DBC($\boldsymbol{\mu}$) is a statistically degraded broadcast channel. As for the 2-user case, according to Lemma 3.1, the boundary of the DPC region of this channel is tangent to the boundary of the DPC region of the original channel at the point $\mathbf{R}^*$.

Furthermore, transmit covariance matrices that achieve $\mathbf{R}^*$ by the DPC scheme in DBC($\boldsymbol{\mu}$) are provided in the following lemma.

Lemma 3.3 *For the case $0 < \mu_1 < \cdots < \mu_K$, the transmit covariance matrices of DBC($\boldsymbol{\mu}$) that achieve the boundary point $\mathbf{R}^*$ as given in (11) by the DPC scheme satisfy the following recursive formulas:*

$$\bar{\Gamma}_k^* = \frac{\mu_k}{2\lambda^*}\left(\sum_{j=k+1}^{K} H_j^T S_j^* H_j + I_t\right)^{-1} - \sum_{j=1}^{k-1} \bar{\Gamma}_j^* - Q_k, \quad k = 1, \ldots, K, \tag{32}$$

and can be expressed as given below:

$$\bar{\Gamma}_k^* = \frac{\mu_k}{2\lambda^*}\left(\sum_{j=k+1}^{K} H_j^T S_j^* H_j + I_t\right)^{-1} - \frac{\mu_k}{2\lambda^*}\left(\sum_{j=k}^{K} H_j^T S_j^* H_j + I_t\right)^{-1} \quad k = 1, \ldots, K, \tag{33}$$

where S_k^* and λ^* are the optimal solutions to the optimization problem (12). Furthermore, for the case $0 < \mu_1 < \cdots < \mu_l = \cdots = \mu_{l+m} < \cdots < \mu_K$, these covariance matrices achieve $\mathbf{R}^{\sigma_1}$ as given in (17)-(18) by the DPC where σ_1 is the identity permutation on $\{l, \ldots, l+m\}$.

Proof: Starting from $k = 1$ and substituting the expressions for the covariance matrices in the recursive formulas, $\bar{\Gamma}_k^*$ as given in (33) are obtained. First, it is shown that $\bar{\Gamma}_k^*$, $k = 1, \ldots, K$, are feasible and satisfy the sum power constraint P. Note that for $k = 1, \ldots, K$,

$$\left(\sum_{j=k}^{K} H_j^T S_j^* H_j + I_t\right)^{-1} \preceq \left(\sum_{j=k+1}^{K} H_j^T S_j^* H_j + I_t\right)^{-1}.$$

Therefore, $\bar{\Gamma}_k^*$ are positive semi-definite. Moreover, using the identity $I - (I + A)^{-1} = (I + A)^{-1}A$ for any $A \succeq 0$, the expressions for $\bar{\Gamma}_k^*$ can be simplified to,

$$\begin{aligned}
\bar{\Gamma}_k^* &= \frac{\mu_k}{2\lambda^*}\left(\left(\sum_{j=k+1}^{K} H_j^T S_j^* H_j + I_t\right)^{-1} - I_t\right) + \frac{\mu_k}{2\lambda^*}\left(I_t - \left(\sum_{j=k}^{K} H_j^T S_j^* H_j + I_t\right)^{-1}\right) \\
&= \frac{\mu_k}{2\lambda^*}\left(\sum_{j=k}^{K} H_j^T S_j^* H_j + I_t\right)^{-1} \sum_{j=k}^{K} H_j^T S_j^* H_j \\
&\quad - \frac{\mu_k}{2\lambda^*}\left(\sum_{j=k+1}^{K} H_j^T S_j^* H_j + I_t\right)^{-1} \sum_{j=k+1}^{K} H_j^T S_j^* H_j,
\end{aligned}$$

where the term $\mu_k/2\lambda^* I_t$ is added to and subtracted from the original expression for $\bar{\Gamma}_k^*$. Adding up $\bar{\Gamma}_k^*$ for $k = 1, \ldots, K$ yields,

$$\begin{aligned}
\sum_{k=1}^{K} \bar{\Gamma}_k^* &= \sum_{k=1}^{K} \frac{\mu_k - \mu_{k-1}}{2\lambda^*} \left(\sum_{i=k}^{K} H_i^T S_i^* H_i + I_t \right)^{-1} \sum_{j=k}^{K} H_j^T S_j^* H_j \\
&= \sum_{j=1}^{K} \sum_{k=1}^{j} \frac{\mu_k - \mu_{k-1}}{2\lambda^*} \left(\sum_{i=k}^{K} H_i^T S_i^* H_i + I_t \right)^{-1} H_j^T S_j^* H_j \\
&= \sum_{j=1}^{K} Q_j H_j^T S_j^* H_j,
\end{aligned}$$

where the second equality is obtained by interchanging the summation over j and k and the third equality follows from definition of Q_k in (31). Hence,

$$\sum_{k=1}^{K} \mathrm{tr}(\bar{\Gamma}_k^*) = \sum_{k=1}^{K} \mathrm{tr}(H_k Q_k H_k^T S_k^*) = \sum_{k=1}^{K} \mathrm{tr}\left((I_r - \frac{1}{\lambda^*} \Phi_k^*) S_k^* \right) = \sum_{k=1}^{K} \mathrm{tr}(S_k^*) = P,$$

where the second equality follows from the optimality conditions given in (15) and definition of Q_k while the third equality follows from the complementary slackness conditions in (16). Therefore, $\bar{\Gamma}_k^*$ as given in (33) satisfy the power constraint P.

Referring to Lemma 3.1, for the case $0 < \mu_1 < \cdots < \mu_K$, $\mathbf{R}^*$ is achievable in DBC($\boldsymbol{\mu}$) by the DPC scheme with an encoding order that is reverse of the decoding order achieving this point in the dual MAC. Hence, starting from user K, user k is encoded after users $k+1, \ldots, K$. On the other hand, according to the DPC rates given in (4), the following rates are achievable in DBC($\boldsymbol{\mu}$) using $\bar{\Gamma}_k^*$ and the mentioned encoding order:

$$\bar{R}_k = \frac{1}{2} \log \frac{\left| \sum_{j=1}^{k} \bar{\Gamma}_j^* + Q_k \right|}{\left| \sum_{j=1}^{k-1} \bar{\Gamma}_j^* + Q_k \right|}, \quad k = 1, \ldots, K.$$

Form the recursive expressions in (32), it is easy to verify the $\bar{R}_k = R_k^*$ for $\mathbf{R}^*$ as specified in (11). Therefore,

$$R_k^* = \frac{1}{2} \log \frac{\left| \sum_{j=1}^{k} \bar{\Gamma}_j^* + Q_k \right|}{\left| \sum_{j=1}^{k-1} \bar{\Gamma}_j^* + Q_k \right|}, \quad k = 1, \ldots, K, \tag{34}$$

and is achievable by the DPC scheme using the covariance matrices $\bar{\Gamma}_k^*$ in DBC($\boldsymbol{\mu}$). Recall that for the case $0 < \mu_1 < \cdots < \mu_l = \cdots = \mu_{l+m} < \cdots < \mu_K$, as is explained in Lemma 2.3, in general, the boundary point $\mathbf{R}^*$ is not necessarily achievable by only successive encoding scheme and further time-sharing may be required. However, following the same line of reasoning as given above, it can be shown that the vertex $\mathbf{R}^{\sigma_1}$ as given in (17)-(18) is achievable by the DPC scheme using the covariance matrices $\bar{\Gamma}_k^*$ in DBC($\boldsymbol{\mu}$). □

The fact that the capacity region of DBC($\boldsymbol{\mu}$) contains the capacity region of the original channel follows directly from the inequalities $H_k^T Q_k H_k \preceq I_r$, exactly in the same manner as for the two user case. Finally by showing that the point $\mathbf{R}^*$ also lies on the boundary of the capacity region of DBC($\boldsymbol{\mu}$), the converse proof is completed. This is proven by contradiction in the following.

As was mentioned earlier, DBC($\boldsymbol{\mu}$) is a degraded broadcast channel and its capacity region is equal to the capacity region of a physically degraded broadcast channel given as below,

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x} + \mathbf{z}_1, \\ \mathbf{y}_k &= \mathbf{y}_{k-1} + \mathbf{z'}_k, \quad k = 2, \ldots, K, \end{aligned}$$

where $\mathbf{z}_1 \in \mathbb{R}^t$ and $\mathbf{z'}_k \in \mathbb{R}^t$ for $k = 2, \ldots, K$ are independent white Gaussian noise vectors with covariance matrices Q_1 and $Q_k - Q_{k-1}$, respectively. For capacity analysis, this physically degraded version of DBC($\boldsymbol{\mu}$) is considered.

Assume $\mathbf{R}^*$ is not on the boundary of the capacity region of DBC($\boldsymbol{\mu}$). Therefore, there must exist a rate-tuple $\mathbf{R}$ and an arbitrary small $\delta > 0$ such that $R_k^* + 2\delta \leq R_k$ for $k = 1, \ldots, K$. Consider $C(e^{nR_1}, e^{nR_2}, \ldots, e^{nR_K}, n)$, any sequence of n block-length codes for DBC($\boldsymbol{\mu}$) with rates $(R_1, \ldots, R_K)$ such that the average probability of decoding error, $P_e^{(n)}$, goes to zero as $n \to \infty$. By Fano's inequality, for sufficiently large n, the following inequalities hold:

$$R_k^* + \delta \leq \frac{1}{n} I(W_k; \mathbf{y}_k^n | W_{k+1}, \ldots, W_K) \quad k = 1, \ldots, K. \tag{35}$$

In Lemma 3.4 at the end of this section, it is shown that for sufficiently small $\epsilon_k > 0$, the inequality,

$$\frac{1}{n} h(\mathbf{y}_k^n | W_{k+1}, W_{k+2}, \ldots, W_K) \geq \frac{1}{2} \log(2\pi e)^t \left| \sum_{j=1}^{k} \bar{\Gamma}_j^* + Q_k \right| + \epsilon_k, \tag{36}$$

holds for $k = 1, \ldots, K$ in the case $0 < \mu_1 < \cdots < \mu_K$ and it holds for $k \in \{1, \ldots, K\} \setminus \{l, \ldots, l+m-1\}$ in the case $0 < \mu_1 < \cdots < \mu_l = \cdots = \mu_{l+m} < \cdots < \mu_K$. As a result, for $k = K$, it follows from (36) that,

$$\frac{1}{n} h(\mathbf{y}_K^n) \geq \frac{1}{2} \log(2\pi e)^t \left| \sum_{j=1}^{K} \bar{\Gamma}_j^* + Q_K \right| + \epsilon_K.$$

However, $\mathbf{y}_K^n = \mathbf{x}^n + \mathbf{z}_K^n$, where $\mathbf{z}_K^n$ consists of i.i.d. Gaussian noise vectors with covariance matrix Q_K for each transmission and there is an average power constraint nP on the transmitted codewords, i.e., $\mathbb{E}(\mathbf{x}^{nT} \mathbf{x}^n) \leq nP$. Therefore, i.i.d. Gaussian $\mathbf{x}(i)$ for $i = 1, \ldots, n$ that satisfy the water-filling conditions maximize $h(\mathbf{y}_K^n)$. Note that from Lemma 3.3, $\sum_{k=1}^{K} \operatorname{tr}(\bar{\Gamma}_k^*) = P$ and from the recursive formulas in (32) for $k = K$, it follows that,

$$\sum_{j=1}^{K} \bar{\Gamma}_j^* + Q_K = \frac{\mu_K}{2\lambda^*} I_t.$$

Therefore, i.i.d. Gaussian $\mathbf{x}(i)$, for $i = 1, \ldots, n$ with covariance matrix $\mathbb{E}(\mathbf{x}(i)\mathbf{x}^T(i)) = \sum_{k=1}^{K} \bar{\Gamma}_k^*$ satisfy the water-filling conditions. Consequently, $\frac{1}{n}h(\mathbf{y}_K^n)$ is bounded from above as given below:

$$\frac{1}{n}h(\mathbf{y}_K^n) \leq \frac{1}{2}\log(2\pi e)^t \left| \sum_{k=1}^{K} \bar{\Gamma}_k^* + Q_K \right|,$$

which contradicts the previous inequality. Therefore, the point $\mathbf{R}^*$ must be on the boundary of the capacity region of DBC($\boldsymbol{\mu}$).

Lemma 3.4 *For the case $0 < \mu_1 < \cdots < \mu_K$, the following inequalities hold for $k = 1, \ldots, K$:*

$$\frac{1}{n}h(\mathbf{y}_k^n | W_{k+1}, W_{k+2}, \ldots, W_K) \geq \frac{1}{2}\log(2\pi e)^t \left| \sum_{j=1}^{k} \bar{\Gamma}_j^* + Q_k \right| + \epsilon_k,$$

where $\epsilon_k > 0$ are arbitrary small positive constants. The aforementioned inequalities hold for $1 \leq k \leq l-1$ and $l+m \leq k \leq K$ for the case $0 < \mu_1 < \cdots < \mu_l = \mu_{l+1} = \cdots = \mu_{l+m} < \cdots < \mu_K$.

Proof: These inequalities are proven by induction on k. First consider the case $0 < \mu_1 < \cdots < \mu_K$. For $k = 1$, from the Fano's inequality in (35), the rate expression for R_1^* in (34) and $h(\mathbf{y}_1^n | W_1, \ldots, W_K) = h(\mathbf{z}_1^n) = \frac{n}{2}\log(2\pi e)^t |Q_1|$, it follows that,

$$\frac{1}{n}h(\mathbf{y}_1^n | W_2, \ldots, W_K) \geq \frac{1}{2}\log(2\pi e)^t |Q_1| + \frac{1}{2}\log\frac{|\bar{\Gamma}_1^* + Q_1|}{|Q_1|} + \delta = \frac{1}{2}\log(2\pi e)^t |\bar{\Gamma}_1^* + Q_1| + \delta,$$

which is the desired inequality with $\epsilon_1 = \delta > 0$. Now assume the inequality holds for $k-1$. Recall that $\mathbf{y}_{k-1}^n$ and $\mathbf{z}'^n_k$ are independent given $W_k, \ldots, W_K$ and $\mathbf{y}_k^n = \mathbf{y}_{k-1}^n + \mathbf{z}'^n_k$, hence, the conditional entropy power inequality can be employed to obtain,

$$\begin{aligned} \exp\left(\frac{2}{nt}h(\mathbf{y}_k^n | W_k, \ldots, W_K)\right) &\geq \exp\left(\frac{2}{nt}h(\mathbf{y}_{k-1}^n | W_k, \ldots, W_K)\right) + \exp\left(\frac{2}{t}h(\mathbf{z}'_k)\right) \\ &\geq 2\pi e \left| \sum_{j=1}^{k-1} \bar{\Gamma}_j^* + Q_{k-1} \right|^{1/t} + 2\pi e \left| Q_k - Q_{k-1} \right|^{1/t} + \epsilon'_{k-1}, \end{aligned}$$

where the second inequality follows from the induction assumption for $k-1$ and sufficiently small $\epsilon'_{k-1} > 0$. However, the recursive expression in (32) for $\bar{\Gamma}_{k-1}^*$ reveals that,

$$\sum_{j=1}^{k-1} \bar{\Gamma}_j^* + Q_{k-1} = \frac{\mu_{k-1}}{2\lambda^*}\left(\sum_{j=k}^{K} H_j^T S_j^* H_j + I_t\right)^{-1}.$$

Therefore, $\sum_{j=1}^{k-1} \bar{\Gamma}_j^* + Q_{k-1}$ and $Q_k - Q_{k-1}$ are scaled versions of each other which yields,

$$\left| \sum_{j=1}^{k-1} \bar{\Gamma}_j^* + Q_{k-1} \right|^{1/t} + |Q_k - Q_{k-1}|^{1/t} = \left| \sum_{j=1}^{k} \bar{\Gamma}_j^* + Q_k \right|^{1/t}.$$

This equality further simplifies the lower bound on $h(\mathbf{y}_k^n|W_k,\dots,W_K)$ as,

$$\frac{1}{n}h(\mathbf{y}_k^n|W_k,\dots,W_K) \geq \frac{1}{2}\log(2\pi e)^t\left|\sum_{j=1}^{k-1}\bar{\Gamma}_j^* + Q_k\right| + \epsilon''_{k-1},$$

for some sufficiently small $\epsilon''_{k-1} > 0$. By employing this lower bound in the Fano's inequality in (35) for user k and replacing the rate expression for R_k^*, the desired inequality is obtained for k:

$$\begin{aligned}
\frac{1}{n}h(\mathbf{y}_k^n|W_{k+1},\dots,W_K) &\geq R_k^* + \frac{1}{n}h(\mathbf{y}_k^n|W_k,\dots,W_K) + \delta \\
&\geq \frac{1}{2}\log\frac{\left|\sum_{j=1}^{k}\bar{\Gamma}_j^* + Q_k\right|}{\left|\sum_{j=1}^{k-1}\bar{\Gamma}_j^* + Q_k\right|} + \frac{1}{2}\log(2\pi e)^t\left|\sum_{j=1}^{k-1}\bar{\Gamma}_j^* + Q_k\right| + \epsilon''_{k-1} + \delta \\
&= \frac{1}{2}\log(2\pi e)^t\left|\sum_{j=1}^{k}\bar{\Gamma}_j^* + Q_k\right| + \epsilon_k.
\end{aligned}$$

Next consider the case $0 < \mu_1 < \cdots < \mu_l = \cdots = \mu_{l+m} < \cdots < \mu_K$. In this case, since $Q_l = \cdots = Q_{l+m}$ in DBC($\boldsymbol{\mu}$), the received vectors $\mathbf{y}_l^n,\dots,\mathbf{y}_{l+m}^n$ are statistically the same. Therefore, for $k = l,\dots,l+m$, the Fano's inequalities in (35) can be written as,

$$R_k^* + \delta \leq \frac{1}{n}I(W_k;\mathbf{y}_k^n|W_{k+1},\dots,W_K) = I(W_k;\mathbf{y}_{l+m}^n|W_{k+1},\dots,W_K).$$

Adding up these inequalities for $k = l,\dots,l+m$ and using the chain rule for mutual information provide the following inequality that will be used subsequently:

$$\sum_{k=l}^{l+m} R_k^* + (m+1)\delta \leq \frac{1}{n}I(W_l,\dots,W_{l+m};\mathbf{y}_{l+m}^n|W_{l+m+1},\dots,W_K). \tag{37}$$

According to Lemma 2.3, for this choice of weights, the point $\mathbf{R}^*$ that is on the boundary of the DPC regions of both DBC($\boldsymbol{\mu}$) and the original channel, lies on the convex-hull of the vertices $\mathbf{R}^{\sigma_i}$ for $i = 1,\dots,(m+1)!$ as given in (17)-(18). Moreover, all these vertices and hence $\mathbf{R}^*$ satisfy the equalities in (19)-(20). Therefore, $R_k^* = R_k^{\sigma_i}$ for $k \in \{1,\dots,K\}\setminus\{l,\dots,l+m\}$ and $\sum_{k=l}^{l+m} R_k^* = \sum_{k=l}^{l+m} R_k^{\sigma_i}$. In particular, these equalities hold for $\mathbf{R}^{\sigma_1}$ where σ_1 is the identity permutation on $\{l,\dots,l+m\}$. On the other hand, in Lemma 3.3 it is shown that the vertex $\mathbf{R}^{\sigma_1}$ is achievable in DBC($\boldsymbol{\mu}$) by using the DPC scheme with covariance matrices $\bar{\Gamma}_k^*$ for $k = 1,\dots,K$ and $R_k^{\sigma_1}$ assures the rate expression in (34). Hence, any boundary point $\mathbf{R}^*$ satisfies,

$$R_k^* = \frac{1}{2}\log\frac{\left|\sum_{j=1}^{k}\bar{\Gamma}_j^* + Q_k\right|}{\left|\sum_{j=1}^{k-1}\bar{\Gamma}_j^* + Q_k\right|} \quad k \in \{1,\dots,K\}\setminus\{l,\dots,l+m\}, \tag{38}$$

$$\sum_{k=l}^{l+m} R_k^* = \sum_{k=l}^{l+m}\frac{1}{2}\log\frac{\left|\sum_{j=1}^{k}\bar{\Gamma}_j^* + Q_k\right|}{\left|\sum_{j=1}^{k-1}\bar{\Gamma}_j^* + Q_k\right|} = \frac{1}{2}\log\frac{\left|\sum_{j=1}^{l+m}\bar{\Gamma}_j^* + Q_{l+m}\right|}{\left|\sum_{j=1}^{l-1}\bar{\Gamma}_j^* + Q_l\right|}, \tag{39}$$

where the second equality in (39) is obtained from the fact that for $\mu_l = \cdots = \mu_{l+m}$, $Q_l = \cdots = Q_{l+m}$. Since the rate expressions for R_k^*, $k = 1, \ldots, l-1$, in (38) are the same as in the case $0 < \mu_1 < \cdots < \mu_K$, identical induction arguments prove the desired inequalities for $k = 1, \ldots, l-1$ in the same way as given before. Now assume the inequality holds for $l-1$, i.e.,

$$\frac{1}{n} h(\mathbf{y}_{l-1}^n | W_l, W_{l+1}, \ldots, W_K) \geq \frac{1}{2} \log(2\pi e)^t \left| \sum_{j=1}^{l-1} \bar{\Gamma}_j^* + Q_{l-1} \right| + \epsilon_{l-1}.$$

In the degraded version of DBC($\boldsymbol{\mu}$), $\mathbf{y}_{l+m}^n = \mathbf{y}_{l-1}^n + \mathbf{z'}_l^n$ since $Q_l = \cdots = Q_{l+m}$. The conditional entropy power inequality can be applied to $\mathbf{y}_{l+m}^n$ to obtain,

$$\begin{aligned} \exp\left(\frac{2}{nt} h(\mathbf{y}_{l+m}^n | W_l, \ldots, W_K)\right) &\geq \exp\left(\frac{2}{nt} h(\mathbf{y}_{l-1}^n | W_l, \ldots, W_K)\right) + \exp\left(\frac{2}{t} h(\mathbf{z}'_l)\right) \\ &\geq 2\pi e \left| \sum_{j=1}^{l-1} \bar{\Gamma}_j^* + Q_{l-1} \right|^{1/t} + 2\pi e \left| Q_l - Q_{l-1} \right|^{1/t} + \epsilon'_{l-1}, \end{aligned}$$

where the second inequality follows from the induction assumption for $k = l-1$ and arbitrary small $\epsilon'_{l-1} > 0$. Again, the two expressions on the right-hand side are scaled versions of each other and simplify further the lower bound on $h(\mathbf{y}_{l+m}^n | W_l, \ldots, W_K)$ as,

$$\frac{1}{n} h(\mathbf{y}_{l+m}^n | W_l, \ldots, W_K) \geq \frac{1}{2} \log(2\pi e)^t \left| \sum_{j=1}^{l-1} \bar{\Gamma}_j^* + Q_l \right| + \epsilon''_{l-1},$$

where $\epsilon''_{l-1} > 0$ is sufficiently small. By employing this lower bound in the inequality (37) and replacing the expression for $\sum_{k=l}^{l+m} R_k^*$ from (39), the desired inequality for $k = l+m$ is obtained as below:

$$\begin{aligned} &\frac{1}{n} h(\mathbf{y}_{l+m}^n | W_{l+m+1}, \ldots, W_K) \\ &\geq \frac{1}{n} h(\mathbf{y}_{l+m}^n | W_l, \ldots, W_K) + \sum_{k=l}^{l+m} R_k^* + (m+1)\delta \\ &\geq \frac{1}{2} \log(2\pi e)^t \left| \sum_{j=1}^{l-1} \bar{\Gamma}_j^* + Q_l \right| + \epsilon''_{l-1} + \frac{1}{2} \log \frac{\left| \sum_{j=1}^{l+m} \bar{\Gamma}_j^* + Q_{l+m} \right|}{\left| \sum_{j=1}^{l-1} \bar{\Gamma}_j^* + Q_l \right|} + (m+1)\delta \\ &= \frac{1}{2} \log(2\pi e)^t \left| \sum_{j=1}^{l+m} \bar{\Gamma}_j^* + Q_{l+m} \right| + \epsilon_{l+m}. \end{aligned}$$

For $k = l+m+1, \ldots, K$, the inequality holds by the same induction steps given previously. $\square$

3.2 Extension to General Convex Constraints on the Transmit Covariance Matrix

In this section, capacity region of the Gaussian MIMO BC under any convex constraint on the transmit covariance matrix is investigated. Consider a norm[2] on the space of symmetric $t \times t$ matrices, $\mathbb{S}^t$, and let $\mathcal{S} \subseteq \mathbb{S}^t$ be a compact (closed and bounded) and convex set with respect to this norm. For the Gaussian MIMO BC defined in (1), instead of the total average power constraint, assume the following constraint is imposed on each codeword of block-length n:

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}(i)\mathbf{x}(i)^T \in \mathcal{S}. \tag{40}$$

Capacity of the Gaussian MIMO BC under these types of transmit covariance matrix constraints is practically interesting. For instance, consider the downlink transmission in a wireless system where each antenna at the base station has its individual average power constraint. These individual power constraints can be imposed by the RF amplifiers connected to each antenna. The problem of determining the data rates supportable for each user in this scenario is particularly important in wireless communications. These per antenna power constraints appropriately lie under the category of the transmit covariance matrix constraints given in (40).

The main objective of this section is to show that the DPC scheme achieves the capacity of the Gaussian MIMO BC under general convex constraints on the transmit covariance matrix. This result was first obtained in [13] directly without any hint of the duality concept. At the first glance, duality may appear to be inappropriate to tackle this problem, since the early versions of the MAC-BC duality introduced in [7], [8] and [10] only observe this duality under total power constraint. However, by using more general duality concepts, in particular the one introduced in [14], optimality of the DPC scheme can be established under these kinds of covariance constraints.

The following theorem formally states the main result of this section.

Theorem 3.2 *The capacity region of the Gaussian MIMO BC in (1) under the transmit covariance matrix constraint in (40), denoted by $\mathcal{C}_{\text{BC}}(\mathcal{S})$, is equal to*

$$\mathcal{R}_{\text{DPC}}(\mathcal{S}) = \boldsymbol{Conv}\left(\bigcup_{\pi,\{\bar{S}_k\}:\bar{S}_k \succeq 0\ \forall k,\ \sum_k \bar{S}_k \in \mathcal{S}} \mathcal{F}(\pi, \{H_k\}, \{\bar{S}_k\}) \right), \tag{41}$$

where π is a permutation on $\{1, \ldots, K\}$, $\bar{S}_k = \mathbb{E}(\mathbf{x}_k\mathbf{x}_k^T)$ is the transmit covariance matrix for user k and the set $\mathcal{F}(\pi, \{H_k\}, \{\bar{S}_k\})$ is defined as in (3).

This theorem is initially established for a particular class of the sets $\mathcal{S}$ that are specified by an arbitrary number of affine constraints on the transmit covariance matrix, $\bar{S} = \sum_k \bar{S}_k$.

[2] Spectral norm is an example. For any matrix $\bar{S} \in \mathbb{S}^t$, $\|\bar{S}\| = \max_i |\lambda_i|$, where λ_i for $i = 1, \ldots, t$ are the eigenvalues of $\bar{S}$.

Then the result is generalized to compact and convex sets in the positive semi-definite cone, $\mathbb{S}^t_+$. Let $\mathcal{M}$ denote a general set in $\mathbb{S}^t_+$ that is specified by a number of affine constraints as given below:

$$\mathcal{M} = \left\{\bar{S} : \bar{S} \succeq 0,\ \mathrm{tr}(A_i\bar{S}) \leq b_i,\ \text{for } i = 1, \ldots, m\right\}, \tag{42}$$

where A_i and b_i, $i = 1, \ldots, m$, are arbitrary number of $t \times t$ symmetric matrices and real numbers, respectively. Clearly, the set $\mathcal{M}$ depends on the matrices A_i and the constants b_i, $i = 1, \ldots, m$, however, this dependency is not included in the notation for simplicity purposes. It is only reasonable to consider the sets $\mathcal{M}$ that are non-empty and bounded, otherwise the corresponding capacity region would be empty or unbounded. Hence, assume A_i and b_i, $i = 1, \ldots, m$, are such that the set $\mathcal{M}$ is non-empty and bounded. Moreover, without loss of generality, it can be assumed that $\sum_{i=1}^m A_i \succeq 0$. Notice that by adding another linear constraint $\mathrm{tr}(A_{m+1}\bar{S}) \leq b_{m+1}$ for $A_{m+1} = -\sum_{i=1}^m A_i$ and $b_{m+1} = \max_{\bar{S}\in\mathcal{M}} \mathrm{tr}(A_{m+1}\bar{S})$, the matrices $\{A_i\}_{i=1}^{m+1}$ satisfy $\sum_{i=1}^{m+1} A_m \succeq 0$ and the set $\mathcal{M}$ is not altered. Also, b_{m+1} is bounded because for matrix $B \in \mathbb{S}^t$, over the bounded set $\mathcal{M}$, $\max_{\bar{S}\in\mathcal{M}} \mathrm{tr}(B\bar{S})$ is also bounded. Since m is arbitrary in definition of this class of covariance constraints specified by $\mathcal{M}$, there is no loss of generality in assuming $\sum_{i=1}^m A_i \succeq 0$. The assumption that $\mathcal{M}$ is non-empty together with $\sum_{i=1}^m A_i \succeq 0$ imply that $\sum_{i=1}^m b_i \geq 0$, since for any positive semi-definite matrix $\bar{S} \in \mathcal{M}$,

$$0 \leq \mathrm{tr}\left(\left(\sum_{i=1}^m A_i\right)\bar{S}\right) \leq \sum_{i=1}^m b_i.$$

To prove Theorem 3.2 for the set $\mathcal{M}$, the notion of MAC-BC duality introduced in [14] is employed. By using the Lagrange dual problem, authors of [14] have established another notion of MAC-BC duality summarized in the following lemma.

Lemma 3.5 *The boundary point $\mathbf{R}^*$ that maximizes $\sum_{k=1}^K \mu_k R_k$ over the DPC region, $\mathcal{R}_{\mathrm{DPC}}(\mathcal{M})$, for some given weights $\mu_1, \ldots, \mu_K \geq 0$, can be obtained by maximizing the same weighted sum of the rates over the capacity region of a Gaussian multiple access channel described as below:*

$$\mathbf{y} = \sum_{k=1}^K H_k^T \mathbf{x}_k + \mathbf{z}.$$

In this dual MAC, the Gaussian noise vector $\mathbf{z} \in \mathbb{R}^r$ has a covariance matrix $S_z \succeq 0$ that is determined by the weights $\mu_1, \ldots, \mu_K$ and the parameters $\{A_i\}, \{b_i\}$ and has the following general form:

$$S_z = \sum_{i=1}^m \alpha_i A_i, \tag{43}$$

for some $\alpha_i \in \mathbb{R}_+$, $i = 1, \ldots, m$, that satisfy,

$$\sum_{i=1}^m \alpha_i b_i \leq \sum_{i=1}^m b_i. \tag{44}$$

Furthermore, the dual MAC has a sum power constraint equal to $\sum_{i=1}^m b_i$.

The notion of MAC-BC duality considered in Section 2.1 is different from the one introduced in Lemma 3.5 in the sense that in the latter, a specific MAC is defined for each boundary point of the DPC region, whereas in the former, a MAC is defined for the whole DPC region. In fact, in the duality notion of [14], the boundary of the capacity region of each dual channel is tangent to the boundary of the DPC region at the point for which the dual channel is defined. However, the capacity region of the dual MAC corresponding to that point may not be equal to the DPC region. It should be mentioned that the duality in [14] has been established originally for the per antenna power constraints. Although the derivation there does not explicitly include the types of the constraints given by the set $\mathcal{M}$, however, by some minor modifications, the duality can be easily extended to this larger class of linear constraints on the transmit covariance matrix. The only subtle point that requires some attention is the choice of $\{A_i\}$ and $\{b_i\}$ that define the set $\mathcal{M}$. For improper choice of $\{A_i\}$ and $\{b_i\}$, the set $\mathcal{M}$ may be empty or unbounded. Furthermore, there may not exist a noise covariance matrix that satisfy (43) and (44), and finally the power constraint in the dual channel, $\sum_{i=1}^{m} b_i$, may be negative. Nevertheless, for the non-empty and bounded sets $\mathcal{M}$ considered in this section, $\sum_{i=1}^{m} A_i \succeq 0$, therefore (43) and (44) at least has a solution for $\alpha_i = 1$, $i = 1, \ldots, m$, and $\sum_{i=1}^{m} b_i \geq 0$.

The same approach used in Section 3 is employed to show that $\mathcal{C}_{\mathrm{BC}}(\mathcal{M})$ is equal to $\mathcal{R}_{\mathrm{DPC}}(\mathcal{M})$. First, each boundary point of $\mathcal{R}_{\mathrm{DPC}}(\mathcal{M})$ is characterized by maximizing $\sum_{k=1}^{K} \mu_k R_k$ for some weights $\mu_1, \ldots, \mu_K \geq 0$. Then for each boundary point, a degraded broadcast channel with the usual total average power constraint is constructed. It is shown that the same point lies on the boundary of the DPC region and the capacity region of this degraded channel. Finally, it is shown that the capacity region of this degraded channel contains $\mathcal{C}_{\mathrm{BC}}(\mathcal{M})$, hence, the boundary point of $\mathcal{R}_{\mathrm{DPC}}(\mathcal{M})$ lies on the boundary of $\mathcal{C}_{\mathrm{BC}}(\mathcal{M})$. By the same arguments, all the boundary points of the DPC region lie on the boundary of the capacity region and two regions are equal.

To mainly focus on the key steps of the proof and avoid getting into unnecessary details, the proof is given for the case where the large channel matrix, $\begin{bmatrix} H_1^T & H_2^T & \cdots & H_K^T \end{bmatrix}^T$ is full column-rank. As it is shown in the following, the dual MAC corresponding to each boundary point of the DPC region of such a broadcast channel has positive definite noise covariance matrix, $S_z \succ 0$. This assumption is only made to simplify the proof, however, the proof can be easily extended to general rank deficient channel matrices by some additional steps required for handling the channel and the noise covariance matrix singularities.

As the first step, the boundary point $\mathbf{R}^*$ that maximizes $\sum_{k=1}^{K} \mu_k R_k$ over $\mathcal{R}_{\mathrm{DPC}}(\mathcal{M})$ for some weights $\mu_1, \ldots, \mu_K \geq 0$ is characterized. As was mentioned in Section 3, it is sufficient to only consider the positive weights $\mu_1, \ldots, \mu_K > 0$. According to Lemma 3.5, this maximization can be performed over the capacity region of the dual MAC. Similar to the dual MAC defined in (6), the dual multiple access channel of Lemma 3.5 has a sum power constraint while in contrast, this channel does not have the identity noise covariance matrix structure. However, the noise at the receiver of this channel can be whitened to transform it into the form of the MAC defined in (6). It is easy to verify that the noise covariance matrix, S_z, of the dual MAC in Lemma 3.5 is positive definite for weights $\mu_1, \ldots, \mu_K > 0$, given that the matrix $\begin{bmatrix} H_1^T & \cdots & H_K^T \end{bmatrix}^T$ is full column-rank. This can be verified by contradiction. Assume

S_z is not positive definite, therefore, there exists $\mathbf{w} \in \mathbb{R}^t$ such that $\mathbf{w}^T S_z \mathbf{w} = 0$. However, $\mathbf{w}$ must be in the null-space of $\begin{bmatrix} H_1^T & \cdots & H_K^T \end{bmatrix}^T$ and $\mathbf{w}^T \begin{bmatrix} H_1^T & \cdots & H_K^T \end{bmatrix}$ must be equal to all zero vector. Otherwise, in the dual MAC, the non-zero element of $\mathbf{w}^T \begin{bmatrix} H_1^T & \cdots & H_K^T \end{bmatrix}$ will have infinite signal to noise ratio and the corresponding user can achieve arbitrary large rates. Since $\mu_k > 0$ for all k, this will make $\sum_k \mu_k R_k$ unbounded above. On the other hand, $\begin{bmatrix} H_1^T & \cdots & H_K^T \end{bmatrix}^T$ is full column-rank and has empty null-space. Therefore, $\mathbf{w}$ cannot be in the null-space of $\begin{bmatrix} H_1^T & \cdots & H_K^T \end{bmatrix}^T$ and S_z must be positive definite.

Given $S_z \succ 0$, the noise at the receiver of the dual MAC in Lemma 3.5 can be whitened by multiplying the output vector by $S_z^{-1/2}$, where $S_z^{-1/2}$ is the square root matrix of S_z^{-1}. This whitening process transforms the channel into the form of the dual MAC defined in (6) with channel matrices $\tilde{H}_k^T = S_z^{-1/2} H_k^T$ for $k = 1, \ldots, K$ and sum power constraint $\sum_{i=1}^m b_i$, and does not alter the capacity region. Therefore, Lemma 2.3 can be employed for this equivalent channel to characterize the boundary point $\mathbf{R}^*$ of $\mathcal{R}_{\text{DPC}}(\mathcal{M})$ that maximizes $\sum_{k=1}^K \mu_k R_k$.

The next couple of steps are exactly the same as in Section 3 and 3.1. For the boundary point $\mathbf{R}^*$ of $\mathcal{R}_{\text{DPC}}(\mathcal{M})$, as is characterized in Lemma 2.3 for the channel matrices $\tilde{H}_k^T = S_z^{-1/2} H_k^T$, $k = 1, \ldots, K$, define the degraded broadcast channel DBC($\boldsymbol{\mu}$) exactly as before. Assume for user k, this channel has the identity channel matrix and the noise covariance matrix $\tilde{Q}_k$ that is given by the same expression in (31) except with all the terms related to the channel matrices H_k^T replaced by the corresponding terms for $\tilde{H}_k^T$. Consistent with the sum power constraint of the dual MAC, assume this channel has a total average power constraint $\sum_{i=1}^m b_i$. Lemma 3.1 holds immediately and $\mathbf{R}^*$ lies on the boundary of the DPC region of DBC($\boldsymbol{\mu}$). Also, following the same line of reasoning as before, it can be shown that $\mathbf{R}^*$ lies on the boundary of the capacity region of DBC($\boldsymbol{\mu}$). Therefore, the only remaining step is to show that the capacity region of the degraded broadcast channel, DBC($\boldsymbol{\mu}$), contains the capacity region of the original channel, $\mathcal{C}_{\text{BC}}(\mathcal{M})$. Recall that DBC($\boldsymbol{\mu}$) has the ordinary total average power constraint while the original channel is under the covariance constraint of the form given in (40). The following lemma proves this claim and complete the proof of Theorem 3.2 for the class of covariance constraints specified by the set $\mathcal{M}$.

Lemma 3.6 *Capacity region of DBC($\boldsymbol{\mu}$) contains $\mathcal{C}_{\text{BC}}(\mathcal{M})$.*

Proof: To prove $\mathcal{C}_{\text{BC}}(\mathcal{M})$ is contained in the capacity region of DBC($\boldsymbol{\mu}$), it is sufficient to show that any code achieving the rate-tuple $\mathbf{R}$ in the original broadcast channel with arbitrary small probability of decoding error, can be used in DBC($\boldsymbol{\mu}$) to achieve the same rates with at least the same probability of decoding error. Consider a code for the original broadcast channel with rates $(R_1, \ldots, R_K)$ with arbitrary small probability of error and codewords denoted by $\mathbf{x}^n$. To use this code in DBC($\boldsymbol{\mu}$), assume each $\mathbf{x}(i)$ for $i = 1, \ldots, n$ is multiplied by $S_z^{1/2}$ prior to transmission in this channel, where $S_z^{1/2}$ is the square root matrix of S_z. After this multiplication, each transmitted codeword has total average power given by,

$$\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}^T(i) S_z \mathbf{x}(i) = \text{tr}(S_z \bar{S}),$$

where $\bar{S} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}(i)\mathbf{x}(i)^T$ is the transmit covariance matrix of that codeword. However, since $\mathbf{x}^n$ is a codeword for the original broadcast channel, it must satisfy the covariance constraint: $\bar{S} \in \mathcal{M}$, i.e.,

$$\mathrm{tr}(A_i\bar{S}) \leq b_i \qquad \text{for } i = 1, \ldots, m.$$

Hence,

$$\mathrm{tr}(S_z\bar{S}) = \sum_{i=1}^{m} \alpha_i \mathrm{tr}(A_i\bar{S}) \leq \sum_{i=1}^{m} \alpha_i b_i \leq \sum_{i=1}^{m} b_i,$$

where the first equality follows form the form of S_z given in (43) and the third inequality is based on (44). In effect, the transmitted codeword in DBC($\boldsymbol{\mu}$) satisfies the total average power constraint for this channel.

On the decoders' side, receiver k of DBC($\boldsymbol{\mu}$) multiplies its received signal by $\tilde{H}_k = H_k S_z^{-1/2}$, adds an i.i.d Gaussian noise vector with covariance matrix $I_r - \tilde{H}_k\tilde{Q}_k\tilde{H}_k^T$ to it and uses the same decoding rule as in the original broadcast channel to decode the transmitted codeword. After these procedures, receiver k obtains the transmitted codeword $\mathbf{x}^n$ passed through the channel matrix $\tilde{H}_k S_z^{1/2} = H_k$ and added to a Gaussian noise vector with covariance matrix $\tilde{H}_k\tilde{Q}_k\tilde{H}_k^T + I_r - \tilde{H}_k\tilde{Q}_k\tilde{H}_k^T = I_r$. Note that the same equalities given in (15) ensures that $\tilde{H}_k\tilde{Q}_k\tilde{H}_k^T \preceq I_r$ for the channel matrices $\tilde{H}_k$. Therefore, the resulting codeword is statistically the same as the one passed through the original channel and the same decoding functions as in the original broadcast channel can be used to decode each user's message with at least the same probability of decoding error. □

In the following, the proof of Theorem 3.2 is extended to any compact and convex set in the cone of $t \times t$ positive semi-definite matrices, $\mathbb{S}_+^t$. The extension makes use of the fact that each closed and convex set can be expressed as the intersection of all closed half-spaces containing it [18]. Recall that any closed half-space in $\mathbb{S}^t$ is expressed by $\mathrm{tr}(AS) \leq b$ for some matrix $A \in \mathbb{S}^t$ and some real number b. Therefore, any compact and convex set $\mathcal{S}$ in $\mathbb{S}_+^t$ can be expressed as the intersection of possibly infinite number of sets $\mathcal{M}_n$ such that for each n, $\mathcal{M}_n$ has the form given in (42) and contains the set $\mathcal{S}$:

$$\mathcal{S} = \bigcap_n \mathcal{M}_n.$$

Note that since $\mathcal{S}$ is bounded, each of the sets $\mathcal{M}_n$ can be chosen as a bounded set, although, they may not be bounded in general.

Now consider a sequence of sets $\{\mathcal{M}_n\}_{n=1}^{\infty}$ such that $\mathcal{M}_1$ has the form given in (42) and contains the set $\mathcal{S}$ and for each n, the set $\mathcal{M}_n$ is obtained from $\mathcal{M}_{n-1}$ by addition of another half-space that contains $\mathcal{S}$. Therefore, $\mathcal{S} = \bigcap_n \mathcal{M}_n$ and

$$\mathcal{M}_m \subseteq \mathcal{M}_n, \qquad \text{for all } m \geq n \text{ and for all } n.$$

The rest of this section is dedicated to prove that $\mathcal{R}_{\mathrm{DPC}}(\cap_n\mathcal{M}_n) = \mathcal{C}_{\mathrm{BC}}(\cap_n\mathcal{M}_n)$. Clearly, $\mathcal{R}_{\mathrm{DPC}}(\cap\mathcal{M}_n) \subseteq \mathcal{C}_{\mathrm{BC}}(\cap_n\mathcal{M}_n)$, thus, it only remains to show that $\mathcal{C}_{\mathrm{BC}}(\cap_n\mathcal{M}_n) \subseteq \mathcal{R}_{\mathrm{DPC}}(\cap_n\mathcal{M}_n)$. For each m, $\bigcap_n \mathcal{M}_n \subseteq \mathcal{M}_m$, therefore, it follows immediately that $\mathcal{C}_{\mathrm{BC}}(\cap_n\mathcal{M}_n) \subseteq \mathcal{C}_{\mathrm{BC}}(\mathcal{M}_m)$

and consequently, $\mathcal{C}_{\rm BC}(\cap_n \mathcal{M}_n) \subseteq \bigcap_n \mathcal{C}_{\rm BC}(\mathcal{M}_n)$. However, from Theorem 3.2 that was proven for the sets $\mathcal{M}_n$ earlier in this section, $\mathcal{R}_{\rm DPC}(\mathcal{M}_n) = \mathcal{C}_{\rm BC}(\mathcal{M}_n)$ for all n which yields,

$$\mathcal{C}_{\rm BC}(\cap_n \mathcal{M}_n) \subseteq \bigcap_n \mathcal{C}_{\rm BC}(\mathcal{M}_n) = \bigcap_n \mathcal{R}_{\rm DPC}(\mathcal{M}_n).$$

In the following it is shown that $\bigcap_n \mathcal{R}_{\rm DPC}(\mathcal{M}_n) = \mathcal{R}_{\rm DPC}(\cap_n \mathcal{M}_n)$ which establishes the relation $\mathcal{C}_{\rm BC}(\cap_n \mathcal{M}_n) \subseteq \mathcal{R}_{\rm DPC}(\cap_n \mathcal{M}_n)$ and completes the proof.

The "tricky" part in showing the equality $\bigcap_n \mathcal{R}_{\rm DPC}(\mathcal{M}_n) = \mathcal{R}_{\rm DPC}(\cap_n \mathcal{M}_n)$ is to verify that $\bigcap_n \mathcal{R}_{\rm DPC}(\mathcal{M}_n) \subseteq \mathcal{R}_{\rm DPC}(\cap_n \mathcal{M}_n)$. Verifying the other direction, $\mathcal{R}_{\rm DPC}(\cap_n \mathcal{M}_n) \subseteq \bigcap_n \mathcal{R}_{\rm DPC}(\mathcal{M}_n)$, is straight forward. First, for any compact and convex set $\mathcal{M} \in \mathbb{S}^t$, let the set $\hat{\mathcal{R}}_{\rm DPC}(\mathcal{M})$ denote the set of all rate-tuples in $\mathcal{R}_{\rm DPC}(\mathcal{M})$ that are achievable by only DPC scheme without any time-sharing, i.e.,

$$\hat{\mathcal{R}}_{\rm DPC}(\mathcal{M}) = \bigcup_{\pi, \{\bar{S}_k\} : \bar{S}_k \succeq 0 \ \forall k, \ \sum_k \bar{S}_k \in \mathcal{M}} \mathcal{F}(\pi, \{H_k\}, \{\bar{S}_k\}). \tag{45}$$

Clearly, $\mathcal{R}_{\rm DPC}(\mathcal{M}) = \operatorname{conv}(\hat{\mathcal{R}}_{\rm DPC}(\mathcal{M}))$. Now, consider a rate-tuple $\mathbf{R}^o \in \bigcap_n \hat{\mathcal{R}}_{\rm DPC}(\mathcal{M}_n)$. $\mathbf{R}^o$ must belong to all the sets $\hat{\mathcal{R}}_{\rm DPC}(\mathcal{M}_n)$ and since each one only contains the rate-tuples achievable by the DPC scheme, for each n, there must exist a permutation π^n on $\{1, \ldots, K\}$ and a set of covariance matrices $(\bar{S}_1^n, \bar{S}_2^n, \ldots, \bar{S}_K^n) \in \left(\mathbb{S}_+^t\right)^K$ such that $\sum_{k=1}^K \bar{S}_k^n \in \mathcal{M}_n$ and these covariance matrices achieve the point $\mathbf{R}^o$ by successive encoding scheme with the order specified by π^n. Given that there are only a finite number of possible permutations on $\{1, \ldots, K\}$, there must exist a permutation π^o and an infinite sub-sequence of permutations, $\{\pi^{n_i}\}_{i=1}^\infty$, that are all equal to π^o. On the other hand, for each n, $\mathcal{M}_n \in \mathcal{M}_1$ and each set of the covariance matrices $(\bar{S}_1^n, \bar{S}_2^n, \ldots, \bar{S}_K^n)$ belongs to the set $\mathcal{K}$ defined as below:

$$\mathcal{K} = \left\{ (\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_K) \in \left(\mathbb{S}_+^t\right)^K : \ \sum_{k=1}^K \bar{S}_k \in \mathcal{M}_1 \right\}.$$

On the vector space $(\mathbb{S}^t)^K$, define a norm $\|(\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_K)\| = \max_k \|\bar{S}_k\|$, where the norm on the right-hand side is the same norm on $\mathbb{S}^t$ considered earlier in this section. Since $\mathcal{M}_1$ is a compact set, $\mathcal{K}$ is also a compact subset of $\left(\mathbb{S}_+^t\right)^K$ under this norm. Therefore, the infinite sub-sequence $\{(\bar{S}_1^{n_i}, \bar{S}_2^{n_i}, \ldots, \bar{S}_K^{n_i})\}_{i=1}^\infty$ must have an infinite sub-sequence that converges to a limiting point $(\bar{S}_1^o, \bar{S}_2^o, \ldots, \bar{S}_K^o) \in \mathcal{K}$, where n_i, $i = 1, 2, \ldots$ are the same indexes for which $\pi^o = \pi^{n_1} = \pi^{n_2} = \cdots$. Since $\mathcal{M}_1 \supseteq \mathcal{M}_2 \supseteq \cdots \supseteq \mathcal{M}_n \supseteq \cdots$, and for each n, $\sum_{k=1}^K \bar{S}_k^n \in \mathcal{M}_n$, it is not hard to show that for the limiting point $(\bar{S}_1^o, \bar{S}_2^o, \ldots, \bar{S}_K^o)$, $\sum_{k=1}^K \bar{S}_k^o \in \bigcap_n \mathcal{M}_n$. Furthermore, in [18], it is shown that the function $\log|I_r + X|$ is continuous for $X \in \mathbb{S}_+^r$ with any norm on the space of symmetric matrices. As a result, the DPC achievable rates obtained by the covariance matrices $(\bar{S}_1^{n_i}, \bar{S}_2^{n_i}, \ldots, \bar{S}_K^{n_i})$ and the encoding order π^o,

$$R_{\pi^o(k)}^o = \frac{1}{2} \log \frac{\left| H_{\pi^o(k)} \left(\sum_{j \geq k} \bar{S}_{\pi^o(j)}^{n_i} \right) H_{\pi^o(k)}^T + I_r \right|}{\left| H_{\pi^o(k)} \left(\sum_{j > k} \bar{S}_{\pi^o(j)}^{n_i} \right) H_{\pi^o(k)}^T + I_r \right|} \qquad k = 1, \ldots, K, \tag{46}$$

converge to the corresponding rate terms for $(\bar{S}_1^o, \bar{S}_2^o, \ldots, \bar{S}_K^o)$ and π^o as the covariance matrices $(\bar{S}_1^{n_i}, \bar{S}_2^{n_i}, \ldots, \bar{S}_K^{n_i})$ converge to $(\bar{S}_1^o, \bar{S}_2^o, \ldots, \bar{S}_K^o)$. Recall that the rates on the left-hand side of the expressions in (46) are equal to R_k^o since $(\bar{S}_1^{n_i}, \bar{S}_2^{n_i}, \ldots, \bar{S}_K^{n_i})$ and π^o achieve $\mathbf{R}^o$. Consequently, $\mathbf{R}^o$ is achievable by $(\bar{S}_1^o, \bar{S}_2^o, \ldots, \bar{S}_K^o)$ with the order π^o and since $\sum_{k=1}^K \bar{S}_k^o \in \bigcap_n \mathcal{M}_n$, $\mathbf{R}^o$ lies in the set $\hat{\mathcal{R}}_{\text{DPC}}(\cap_n \mathcal{M}_n)$.

So far, it was shown that $\bigcap_n \hat{\mathcal{R}}_{\text{DPC}}(\mathcal{M}_n) \subseteq \hat{\mathcal{R}}_{\text{DPC}}(\cap_n \mathcal{M}_n)$. Therefore, the convex-hull of the set $\bigcap_n \hat{\mathcal{R}}_{\text{DPC}}(\mathcal{M}_n)$ is also a subset of the convex-hull of the set $\hat{\mathcal{R}}_{\text{DPC}}(\cap_n \mathcal{M}_n)$ which is equal to $\mathcal{R}_{\text{DPC}}(\cap_n \mathcal{M}_n)$. The following lemma shows $\bigcap_n \mathcal{R}_{\text{DPC}}(\mathcal{M}_n) = \text{conv}\left(\bigcap_n \hat{\mathcal{R}}_{\text{DPC}}(\mathcal{M}_n)\right)$ which in effect proves that $\bigcap_n \mathcal{R}_{\text{DPC}}(\mathcal{M}_n) \subseteq \mathcal{R}_{\text{DPC}}(\cap_n \mathcal{M}_n)$.

Lemma 3.7 *Let the decreasing sets $\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \cdots \supseteq \mathcal{A}_n \supseteq \cdots$ be compact and connected subsets of the d-dimensional Euclidean subspace and let $\bar{\mathcal{A}}_n$ be the convex-hull of the set $\mathcal{A}_n$. Then, it can be shown that*

$$\bigcap_n \bar{\mathcal{A}}_n = conv\left(\bigcap_n \mathcal{A}_n\right).$$

This Lemma is proven in Appendix C.

4 Summary

In this paper, the capacity region of the Gaussian MIMO BC was considered. The DPC achievable rate region for this class of broadcast channels was reviewed. By introducing the dual MAC of a Gaussian MIMO BC and using the MAC-BC duality result, the DPC region was represented alternatively as the capacity region of the dual MAC under sum power constraint. It was shown that the dual representation of the DPC region has several key advantages over the original representation that can be exploited to characterized this region more efficiently. Using the convex optimization techniques, each point on the boundary surface of the DPC region was characterized as the solution to a convex optimization problem.

After characterizing the DPC region, it was proven that this region is the capacity region of the Gaussian MIMO BC under ordinary total average power constraint. In the converse proof, several ideas from the previous works were combined with the duality theory to prove that each point on the boundary of the DPC region lies on the boundary of the capacity region. Finally, by using a more comprehensive notion of MAC-BC duality, the optimality of the DPC scheme was proven under any general convex constraint on the transmit covariance matrix.

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A Proof of Proposition 2.1

In this appendix, it is shown that the capacity region of the dual MAC under sum power constraint, $\mathcal{C}^{sum}_{\text{MAC}}(P)$, as given in (9), is convex.

Proof: Assume the rate-tuples $\mathbf{R}^{(1)}$, $\mathbf{R}^{(2)}$ belong to $\mathcal{C}^{sum}_{\text{MAC}}(P)$. Hence, there must exist two sets of positive semi-definite covariance matrices $\{S_k^{(1)}\}$ and $\{S_k^{(2)}\}$ such that $\sum_k \operatorname{tr}(S_k^{(i)}) \leq P$ and $\mathbf{R}^{(i)} \in \mathcal{G}(\{H_k^T\}, \{S_k^{(i)}\})$ for $i = 1, 2$. For any given $\alpha \in [0, 1]$ and any $J \subseteq \{1, \ldots, K\}$, the following inequalities can be obtained:

$$
\begin{aligned}
\sum_{k \in J} \left(\alpha R_k^{(1)} + \bar{\alpha} R_k^{(2)} \right) &\leq \frac{\alpha}{2} \log \left| \sum_{k \in J} H_k^T S_k^{(1)} H_k + I_t \right| + \frac{\bar{\alpha}}{2} \log \left| \sum_{k \in J} H_k^T S_k^{(2)} H_k + I_t \right| \\
&\leq \frac{1}{2} \log \left| \sum_{k \in J} H_k^T \left(\alpha S_k^{(1)} + \bar{\alpha} S_k^{(2)} \right) H_k + I_t \right|,
\end{aligned}
$$

where $\bar{\alpha} = 1 - \alpha$, the first inequality follows from definition of the set $\mathcal{G}(\{H_k^T\}, \{S_k\})$ and the second one is from concavity of the $\log|.|$ function. As a result, the rate-tuple $\alpha\mathbf{R}^{(1)} + \bar{\alpha}\mathbf{R}^{(2)}$ belongs to the set $\mathcal{G}(\{H_k^T\}, \{\alpha S_k^{(1)} + \bar{\alpha} S_k^{(2)}\})$. Since for the given α, $\sum_k \operatorname{tr}(\alpha S_k^{(1)} + \bar{\alpha} S_k^{(2)}) \leq P$ holds as well, the set of covariance matrices $\{\alpha S_k^{(1)} + \bar{\alpha} S_k^{(2)}\}$ satisfies the power constraint and the region $\mathcal{C}^{sum}_{\text{MAC}}(P)$ is convex. □

B Proof of Lemma 3.1

In this appendix, Lemma 3.1 is proven for the general K-user case. It is shown that the point $\mathbf{R}^*$ as given by Lemma 2.3 maximizes $\sum_{k=1}^{K} \mu_k R_k$ over the DPC region of DBC($\boldsymbol{\mu}$), denoted by $\mathcal{R}_{\mathrm{DPC}}^{\mathrm{DBC}(\boldsymbol{\mu})}(P)$.

Proof: By means of duality, optimization can be performed over the capacity region of the dual MAC of DBC($\boldsymbol{\mu}$) under sum power constraint P. Recall that the MAC-BC duality considered in Section 2.1 applies to channels with spatially white Gaussian noises. Let $Q_k^{-1/2}$ be the symmetric square root matrix of the inverse of Q_k, i.e., $Q_k^{-1/2} = (Q_k^{-1/2})^T$ and $Q_k^{-1/2} Q_k^{-1/2} = Q_k^{-1}$, $k = 1, \dots, K$. The noise at receiver k of DBC($\boldsymbol{\mu}$) can be whitened by multiplying the channel output $\mathbf{y}_k$ by $Q_k^{-1/2}$. This process results in an equivalent broadcast channel with channel matrix $Q_k^{-1/2}$ and receiver's white Gaussian noise $\mathbf{z}_k$ with covariance matrix I_t for user k. Note that this transformation whitens the noises and does not affect the capacity region nor the DPC region of DBC($\boldsymbol{\mu}$). Therefore, the dual MAC of DBC($\boldsymbol{\mu}$) has channel matrices $Q_k^{-1/2}$ for user k and white Gaussian noise $\mathbf{z}$ with covariance matrix I_t, and is given by,

$$\mathbf{y} = \sum_{k=1}^{K} Q_k^{-1/2} \mathbf{x}_k + \mathbf{z}. \tag{47}$$

Figure 6 depicts DBC($\boldsymbol{\mu}$) together with its dual MAC for $K = 2$ users. By duality, the DPC region of DBC($\boldsymbol{\mu}$) is equal to the capacity region of its dual MAC given in (47) under sum power constraint P. Hence, the weighted sum rate maximization can be performed over the capacity region of this dual MAC. Let Γ_k denote the transmit covariance matrix of user k in this dual channel for $k = 1, \dots, K$. For a given set of covariance matrices $\{\Gamma_k\}$ and for all permutations σ_i, $i = 1, \dots, (m+1)!$, on the set $\mathcal{I} = \{l, \dots, l+m\}$, define the rate-tuple $\tilde{\mathbf{R}}^{\sigma_i}$ as,

$$\begin{aligned}
\tilde{R}_k^{\sigma_i} &= \frac{1}{2} \log \frac{\left| \sum_{j=k}^{K} Q_j^{-1/2} \Gamma_j Q_j^{-1/2} + I_t \right|}{\left| \sum_{j=k+1}^{K} Q_j^{-1/2} \Gamma_j Q_j^{-1/2} + I_t \right|} & k \in \{1, \dots, K\} \setminus \mathcal{I}, \\
\tilde{R}_{\sigma_i(k)}^{\sigma_i} &= \frac{1}{2} \log \frac{\left| \sum_{j=k}^{l+m} Q_{\sigma_i(j)}^{-1/2} \Gamma_{\sigma_i(j)} Q_{\sigma_i(j)}^{-1/2} + \sum_{j=l+m+1}^{K} Q_j^{-1/2} \Gamma_j Q_j^{-1/2} + I_t \right|}{\left| \sum_{j=k+1}^{l+m} Q_{\sigma_i(j)}^{-1/2} \Gamma_{\sigma_i(j)} Q_{\sigma_i(j)}^{-1/2} + \sum_{j=l+m+1}^{K} Q_j^{-1/2} \Gamma_j Q_j^{-1/2} + I_t \right|} & k \in \mathcal{I}.
\end{aligned}$$

Also let σ_1 be the identity permutation, i.e., $\sigma_1(k) = k$ for all $k \in \mathcal{I}$. Following the same argument given in the proof of Lemma 2.3, for the case $0 < \mu_1 < \dots < \mu_K$, $\tilde{\mathbf{R}}^{\sigma_1}$ maximizes $\sum_{k=1}^{K} \mu_k R_k$ over the DPC region of DBC($\boldsymbol{\mu}$) for transmit covariance matrices Γ_k^*,

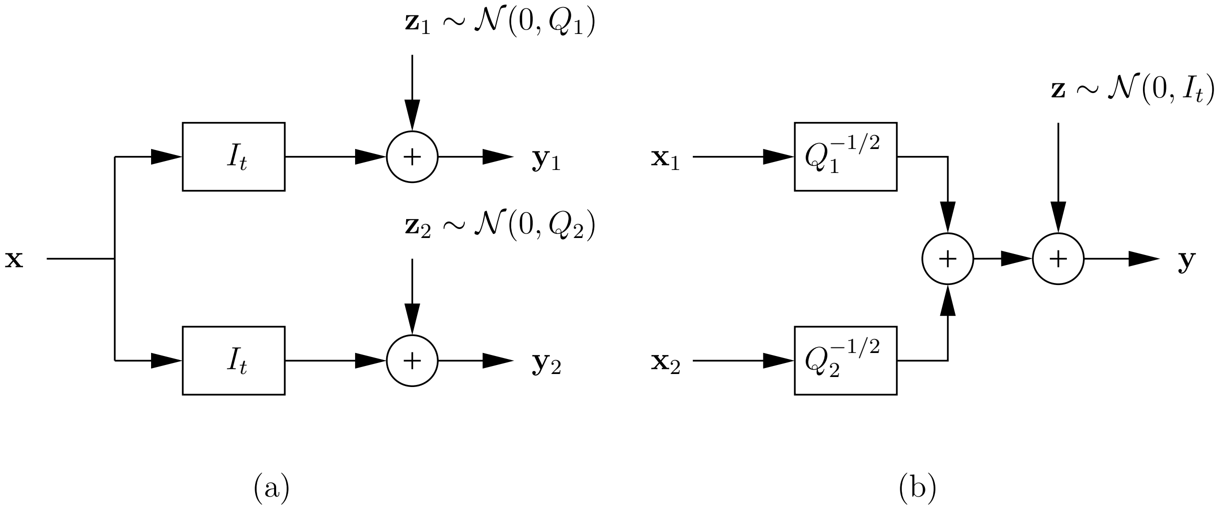

Figure 6: Degraded Gaussian MIMO BC DBC($\boldsymbol{\mu}$) in (a) and its dual multiple access channel in (b).

$k = 1, \ldots, K$, that are the optimal solutions to the following optimization problem:

$$\begin{aligned} \text{Maximize} \quad & \sum_{k=1}^{K} (\mu_k - \mu_{k-1}) \frac{1}{2} \log \left| \sum_{j=k}^{K} Q_j^{-1/2} \Gamma_j Q_j^{-1/2} + I_t \right| \\ \text{Subject to} \quad & \sum_{k=1}^{K} \operatorname{tr}(\Gamma_k) \leq P, \\ & \Gamma_k \succeq 0 \quad k = 1, \ldots, K. \end{aligned} \tag{48}$$

This optimization problem is exactly the same as the one given in (12) except the channel matrices H_k^T of (6) are replaced by the channel matrices of (47), $Q_k^{-1/2}$. Hence, the optimal solution must satisfy the KKT conditions of Lemma 2.3 with H_k^T replaced by $Q_k^{-1/2}$ and any solution that satisfies these KKT conditions is optimal. Let γ and Ψ_k for $k = 1, \ldots, K$ be the dual variables associated with the sum power and the positive semi-definite constraints for the optimization problem (48). Set $\gamma^* = \lambda^*$, $\Psi_k^* = \mathbf{0}$ for all k and

$$\Gamma_k^* = Q_k^{1/2} H_k^T S_k^* H_k Q_k^{1/2} \qquad k = 1, \ldots, K, \tag{49}$$

where S_k^* and λ^* are the primal and the dual optimal solutions of (12). In the following, it is shown that this specific choice of the primal and dual variables satisfies the KKT conditions for the optimization problem in (48) and hence is optimal. Clearly $\Gamma_k^* \succeq 0$ for all k and $\gamma^* > 0$. Also note that,

$$\sum_{k=1}^{K} \operatorname{tr}(\Gamma_k^*) = \sum_{k=1}^{K} \operatorname{tr}(H_k Q_k H_k^T S_k^*) = \sum_{k=1}^{K} \operatorname{tr}\left((I_r - \frac{1}{\lambda^*} \Phi_k^*) S_k^* \right) = \sum_{k=1}^{K} \operatorname{tr}(S_k^*) = P,$$

where the second equality follows from the optimality conditions given in (15) and definition of Q_k while the third equality follows from the complementary slackness conditions in (16).

Hence, Γ_k^*, Ψ_k^* for $k = 1, \ldots, K$ and γ^* as defined are feasible and satisfy the complementary slackness conditions, i.e., $\mathrm{tr}(\Gamma_k^* \Psi_k^*) = 0$ for all k. It only remains to show that the derivatives of the Lagrangian with respect to Γ_k, $k = 1, \ldots, K$, are equal to zero at these given values. In other words, they satisfy the equation (15) counterparts for the MAC in (47):

$$Q_k^{-1/2} \sum_{j=1}^{k} (\mu_j - \mu_{j-1}) \frac{1}{2} \left(\sum_{i=j}^{K} Q_i^{-1/2} \Gamma_i^* Q_i^{-1/2} + I_t \right)^{-1} Q_k^{-1/2} + \Psi_k^* - \gamma^* I_r = \mathbf{0},$$

for $k = 1, \ldots, K$. These equations are obtained from equations (15) by replacing H_k^T with $Q_k^{-1/2}$. Using the definition of Q_k, it is not hard to show that the chosen values for Γ_k^*, Ψ_k^* and γ^* satisfy these equations and are optimal. By substituting the optimal Γ_k^* from (49) in the expression for $\tilde{\mathbf{R}}^{\sigma_1}$, the expression for the optimal rate-tuple $\mathbf{R}^*$ given in (11) is obtained. Therefore, $\mathbf{R}^*$ is on the boundary of the DPC region of DBC($\boldsymbol{\mu}$). Similarly, for the case $0 < \mu_1 < \cdots < \mu_l = \cdots = \mu_{l+m} < \cdots < \mu_K$, each point on the convex-hull of the vertices $\tilde{\mathbf{R}}^{\sigma_i}$, $i = 1, \ldots, (m+1)!$, maximizes $\sum_{k=1}^{K} \mu_k R_k$ over the DPC region of DBC($\boldsymbol{\mu}$) for the optimal transmit covariance matrices Γ_k^* that are obtained from (48). By the same arguments used for $\tilde{\mathbf{R}}^{\sigma_1}$, it can be shown that the boundary point $\tilde{\mathbf{R}}^{\sigma_i}$ of $\mathcal{R}_{\mathrm{DPC}}^{\mathrm{DBC}(\boldsymbol{\mu})}(P)$ coincides with the vertex $\mathbf{R}^{\sigma_i}$ of $\mathcal{R}_{\mathrm{DPC}}(P)$ given in (17)-(18) for $i = 1, \ldots, (m+1)!$. Therefore, the convex-hull of these vertices are the same and the DPC regions of the original channel and DBC($\boldsymbol{\mu}$) both have this surface in common. Recall that the same weights $\mu_1, \ldots, \mu_K$ are used to find the boundary point or surface of both $\mathcal{R}_{\mathrm{DPC}}^{\mathrm{DBC}(\boldsymbol{\mu})}(P)$ and $\mathcal{R}_{\mathrm{DPC}}(P)$, hence, the two boundaries are tangent at this boundary point or surface. □

C Proof of Lemma 3.7

In this appendix, it is shown that for the decreasing, compact and connected subsets of the d-dimensional Euclidean subspace, $\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \cdots \supseteq \mathcal{A}_n \supseteq \cdots$,

$$\bigcap_n \bar{\mathcal{A}}_n = \text{conv}\left(\bigcap_n \mathcal{A}_n\right),$$

where $\bar{\mathcal{A}}_n$ denotes the convex-hull of the set $\mathcal{A}_n$.

Proof: It is easy to show that $\text{conv}\left(\bigcap_n \mathcal{A}_n\right) \subseteq \bigcap_n \bar{\mathcal{A}}_n$. In the following, it is proven that, $\bigcap_n \bar{\mathcal{A}}_n \subseteq \text{conv}\left(\bigcap_n \mathcal{A}_n\right)$. Consider a point $\mathbf{p} \in \bigcap_n \bar{\mathcal{A}}_n$. From the Carathéodory theorem [17], the point $\mathbf{p}$ in the convex-hull of the connected and compact set $\mathcal{A}_n$ can be represented as a convex combination of at most d points in the set $\mathcal{A}_n$. Therefore, for each n, there exist d points $\mathbf{b}_1^n, \mathbf{b}_2^n, \ldots, \mathbf{b}_d^n \in \mathcal{A}_n$ and a vector $\boldsymbol{\alpha}^n = (\alpha_1^n, \ldots, \alpha_d^n) \in \mathbb{R}_+^d$ such that $\sum_{i=1}^d \alpha_i^n = 1$ and $\mathbf{p} = \sum_{i=1}^d \alpha_i^n \mathbf{b}_i^n$. Note that some of the α_i^n may be zero. Since for each n, $\mathcal{A}_n \subseteq \mathcal{A}_1$, $(\mathbf{b}_1^n, \mathbf{b}_2^n, \ldots, \mathbf{b}_d^n) \in \mathcal{A}_1^d$ and they form an infinite sequence in the compact set $\mathcal{A}_1^d$. Hence, there exists a sub-sequence $\{(\mathbf{b}_1^{n_i}, \mathbf{b}_2^{n_i}, \ldots, \mathbf{b}_d^{n_i})\}_{i=1}^\infty$ that converges to a point $(\mathbf{b}_1^o, \mathbf{b}_2^o, \ldots, \mathbf{b}_d^o)$ and obviously each of the points $\mathbf{b}_i^o$ must be in the set $\bigcap_n \mathcal{A}_n$ for $i = 1, \ldots, d$. Recall that the vectors $\boldsymbol{\alpha}^n$ also forms an infinite sequence in the compact set $[0,1]^d$. Therefore, the sub-sequence $\boldsymbol{\alpha}^{n_i}$ for indexes n_i, $i = 1, 2, \ldots$, as specified before, has a sub-sequence that converges to a limiting point $\boldsymbol{\alpha}^o \in [0,1]^d$. Since for each n_i, $\sum_{j=1}^d \alpha_j^{n_i} = 1$ and $\mathbf{p} = \sum_{j=1}^d \alpha_j^{n_i} \mathbf{b}_j^{n_i}$, these equalities must hold for the limiting points as well, i.e.,

$$\sum_{j=1}^d \alpha_j^o = 1,$$

$$\mathbf{p} = \sum_{j=1}^d \alpha_j^o \mathbf{b}_j^o.$$

Hence, $\mathbf{p}$ lies in the convex-hull of the set $\bigcap_n \mathcal{A}_n$. □