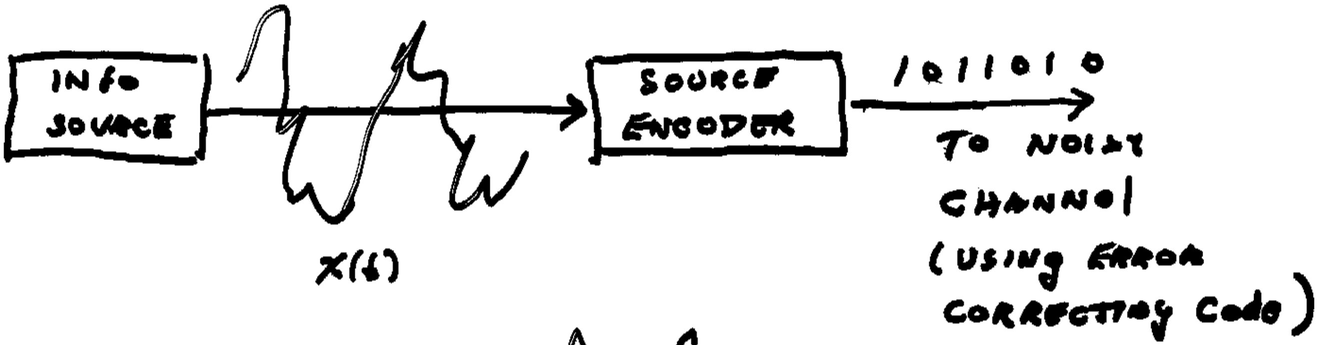
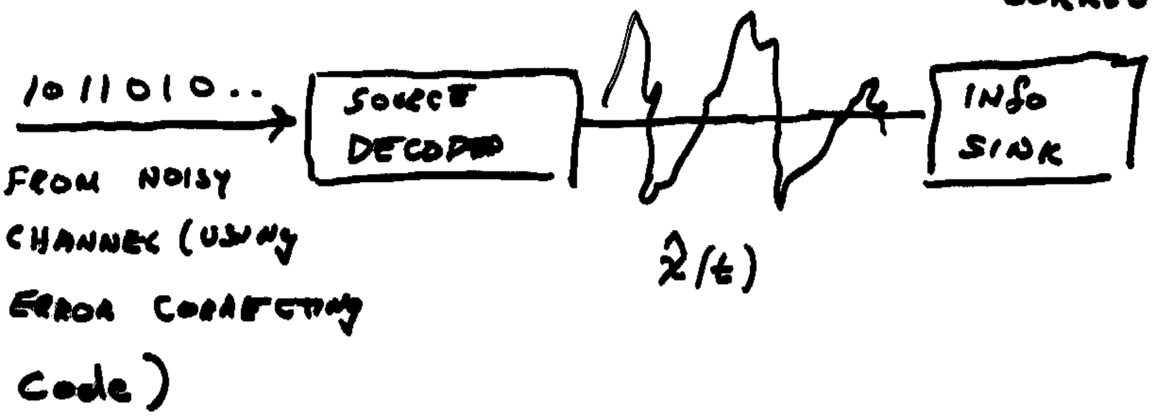
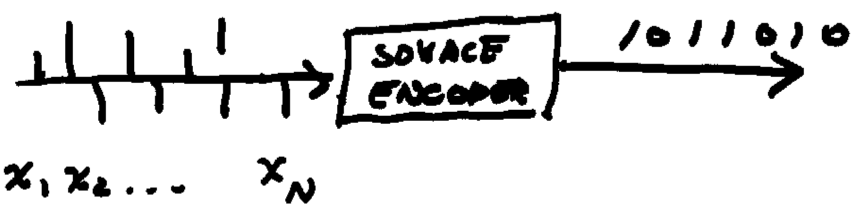
CODING WITH DISTORTION

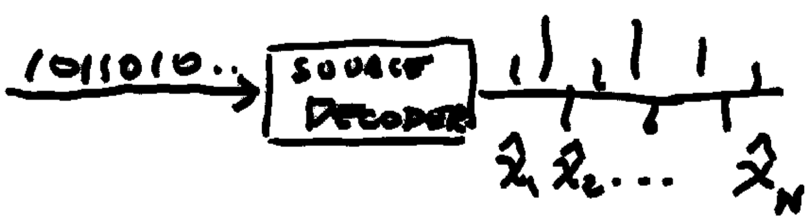




$$\varepsilon^2 = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} E\left(x(t)-\hat{x}(t)\right)^2 dt = \text{M.S.E.}$$

IF signals are bandlimited, ONE CAN SAMPLE AT NYQUIST RATE AND CONVERT CONTINUOUS-TIME Problem TO discrete-TIME problem





$$\varepsilon^2 = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} E\left(x_i-\hat{x}_i\right)^2$$

DISCRETE - TIME GAUSSIAN SOURCE

Let source produce i.i.d Gaussian samples $x_1 x_2 \ldots$

where $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$

Let source encoder produce a sequence of binary digits at a rate of R binary digits/source symbol.

Let the source decoder produce the sequence $\hat{x}_1, \hat{x}_2, \ldots \hat{x}_i \ldots$ such that the mean-squared error between $\{x_i\}_i$ and $\{\hat{x}_i\}_i$ is ε^2.

Then one can prove that for any such system

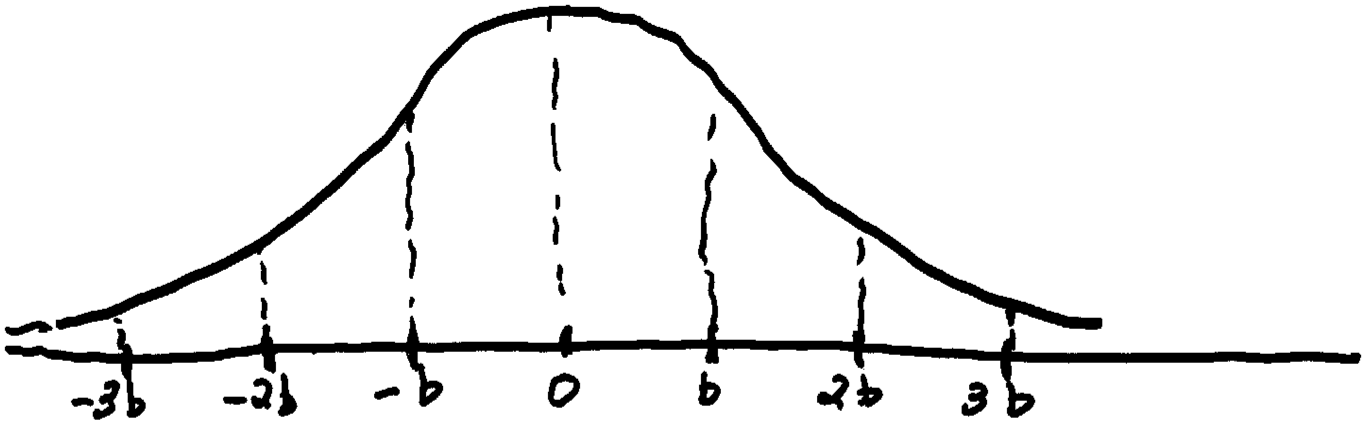
$$R \geq \frac{1}{2}\log_2\left(\frac{\sigma^2}{\varepsilon^2}\right) \quad \text{for} \quad \varepsilon^2 \leq \sigma^2.$$

$(R=0$ for $\varepsilon^2 \geq \sigma^2)$

This is an example of "RATE-DISTORTION THEORY"

SCALAR QUANTIZATION of (GAUSSIAN) samples.

USUAL SCALAR QUANTIZATION (3-BINARY DIGITS/SAMPLE)



ENCODER

$x \leq -3b$	000	$0 < x < b$	100
$- < x \leq -2b$	001	$b < x \leq 2b$	101
$-2b < x \leq -b$	010	$2b < x \leq 3b$	110
$-b < x \leq 0$	011	$3b < x$	111

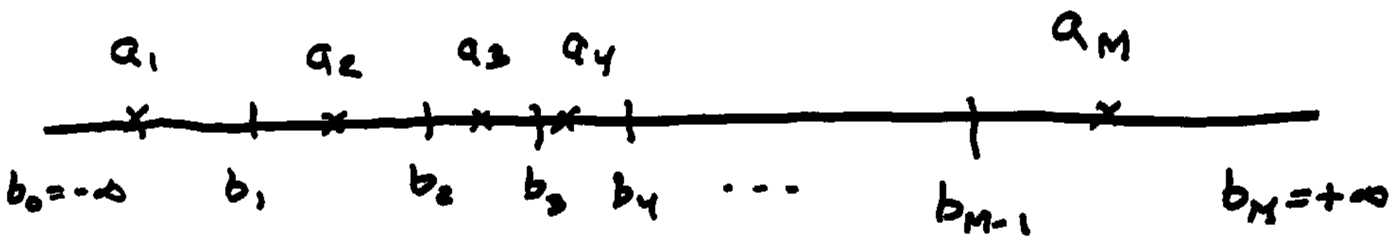
DECODER

000	$-3.5b$	100	$+.5b$
001	$-2.5b$	101	$+1.5b$
010	$-1.5b$	110	$+2.5b$
011	$-.5b$	111	$+3.5b$

OPTIMUM SCALAR QUANTIZER

$$b_{i-1} \le x < b_i \rightarrow \hat{x} = a_i \qquad i = 1, 2, \ldots, M$$

$$b_0 = -\infty, \quad b_M = +\infty$$



OPTIMIZE $\{b_i\}$ and $\{a_i\}$ to MINIMIZE ε^2

$$\varepsilon^2 = \sum_{i=1}^{M} \int_{b_{i-1}}^{b_i} (x - a_i)^2 f_X(x)\, dx$$

$$\frac{\partial \varepsilon^2}{\partial a_j} = 0 \qquad \frac{\partial \varepsilon^2}{\partial b_j} = 0 .$$

USE Leibnitz's Rule

$$\frac{\partial}{\partial t} \int_{a(t)}^{b(t)} f(x,t)\, dx = f(b(t),t) \frac{\partial b(t)}{\partial t}$$

$$- f(a(t),t) \frac{\partial a(t)}{\partial t}$$

$$+ \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)\, dt$$

$$\frac{\partial}{\partial b_j}\left(\sum_{i=1}^{M}\int_{b_{i-1}}^{b_i}(x-a_i)^2 f_X(x)\,dx\right)=$$

$$\frac{\partial}{\partial b_j}\int_{b_{j-1}}^{b_j}(x-a_j)^2 f_X(x)\,dx+\frac{\partial}{\partial b_j}\int_{b_j}^{b_{j+1}}(x-a_{j+1})^2 f_X(x)\,dx$$

$$=(b_j-a_j)^2 f_X(x)\Big|_{x=b_j}-(b_j-a_{j+1})^2 f_X(x)\Big|_{x=b_j}\overset{\text{set}}{=}0$$

$$\cancel{b_j^2}-2a_j b_j+a_j^2=\cancel{b_j^2}-2b_j a_{j+1}+a_{j+1}^2$$

$$2b_j(a_{j+1}-a_j)=a_{j+1}^2-a_j^2$$

$$\boxed{b_j=\frac{a_{j+1}+a_j}{2}}\qquad (\mathrm{I})$$

$$\frac{\partial}{\partial a_j}\left(\sum_{i=1}^{M}\int_{b_{i-1}}^{b_i}(x-a_i)^2 f_X(x)\,dx\right)=-2\int_{b_{j-1}}^{b_j}(x-a_j)\,f_X(x)\,dx\overset{\text{set}}{=}0$$

$$a_j\int_{b_{j-1}}^{b_j}f_X(x)\,dx=\int_{b_{j-1}}^{b_j}x\,f_X(x)\,dx$$

$$\boxed{a_j=\frac{\int_{b_{j-1}}^{b_j}x f_X(x)\,dx}{\int_{b_{j-1}}^{b_j}f_X(x)\,dx}}\qquad (\mathrm{II})$$

NOTE THAT THE $\{b_i\}$ CAN BE FOUND FROM (I) ONCE THE $\{a_i\}$ ARE KNOWN. (THE $\{b_i\}$ ARE THE MIDPOINTS OF THE $\{a_i\}$.)

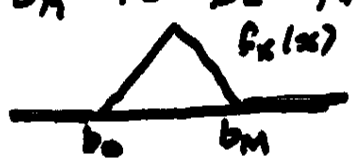
AND THE $\{a_i\}$ CAN BE SOLVED FROM (II) ONCE THE $\{b_i\}$ ARE KNOWN. (THE $\{a_i\}$ ARE THE CENTROIDS OF THE CORRESPONDING REGIONS.)

THUS ONE CAN USE A COMPUTER TO <u>ITERATIVELY</u> SOLVE FOR THE $\{a_i\}$ AND THE $\{b_i\}$.

1. ONE STARTS WITH AN INITIAL GUESS FOR THE $\{b_i\}$.
2. ONE USES (II) TO SOLVE FOR THE $\{a_i\}$.
3. ONE USE (I) TO SOLVE FOR THE $\{b_i\}$
4. ONE REPEATS STEPS 2 AND 3 UNTIL THE $\{a_i\}$ AND THE $\{b_i\}$ "STOP CHANGING."

COMMENTS

1. THIS WORKS FOR ANY $f_X(x)$
2. IF $f_X(x)$ ONLY HAS A <u>FINITE</u> <u>SUPPORT</u> ONE ADJUSTS b_0 & b_M TO BE THE LIMITS OF THE SUPPORT.



3. FOR A GAUSSIAN, ONE NEEDS TO KNOW
$$\int_\alpha^\beta f_X(x)\,dx \quad \text{and} \quad \int_\alpha^\beta x f_X(x)\,dx.$$
(TRUE FOR ANY $f_X(x)$)

$$\int_{\alpha}^{\beta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = Q(\beta) - Q(\alpha)$$

$$\int_{\alpha}^{\beta} x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \dots$$ (integrate by parts)
(or let $y = x^2$)

4. If $M = 2^a$ one could use "a" binary digits to represent the quantized value. However since the quantized values are not necessarily equally likely, one could use a Huffman code to use fewer binary digits (on the average)

5. After the $\{a_i\}$ and $\{b_i\}$ are known, one computes ε^2 from

$$\varepsilon^2 = \sum_{i=1}^{M} \int_{b_{i-1}}^{b_i} (x - a_i)^2 f_X(x) dx$$

6. For $M = 2$ and $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2} x^2/\sigma^2}$ one can easily show that: $b_0 = -\infty$, $b_1 = 0$, $b_2 = +\infty$,

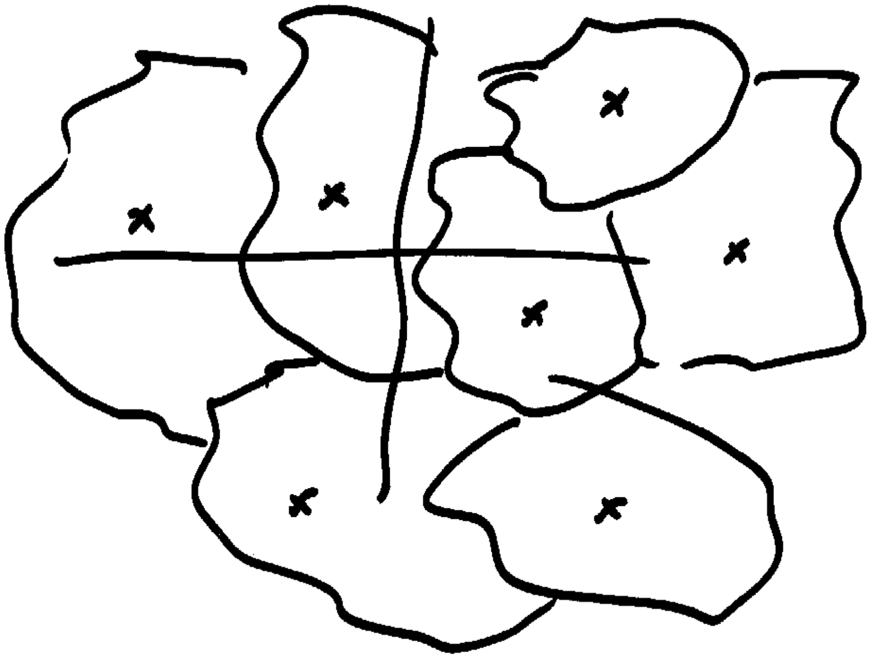
$a_2 = -a_1 = \sqrt{\frac{2\sigma^2}{\pi}}$

$\varepsilon^2 = \left(1 - \frac{2}{\pi}\right)\sigma^2 = .3634\sigma^2$

VECTOR QUANTIZATION

ONE CAN ACHIEVE A SMALLER ε^2 by QUANTIZING SEVERAL SAMPLES AT A TIME.

We would then use REGIONS IN A m-DIMENSIONAL SPACE



THE RATE-DISTORTION FORMULA TELLS US HOW SMALL ε^2 CAN be as $m \to \infty$.

FOR A GAUSSIAN WITH ONE BINARY DIGIT PER SAMPLE, $\varepsilon^2 \geq \frac{\sigma^2}{4} = (0.25)\sigma^2$

ENTROPY, JOINT ENTROPY, CONDITIONAL ENTROPY AND MUTUAL INFORMATION

SINCE $$H(X) = \sum_x P[x] \log \frac{1}{P[x]}$$

THEN $$H(X,Y) = \sum_x \sum_y P[x,y] \log \frac{1}{P[x,y]}$$

BUT $P[x,y] = P[y|x]\,P[x]$ so

$$H(X,Y) = \sum_x \sum_y P[x,y] \left[\log \frac{1}{P[y|x]} + \log \frac{1}{P[x]} \right]$$

$$= \sum_x \sum_y P[x,y] \log \frac{1}{P[y|x]} + \sum_x \underbrace{\left(\sum_y P[x,y] \right)}_{P[x]} \log \frac{1}{P[x]}$$

$$= \underbrace{\sum_x \sum_y P[x,y] \log \frac{1}{P[y|x]}}_{\text{Defined as } H[Y|X]} + \underbrace{\sum_x P[x] \log \frac{1}{P[x]}}_{H(X)}$$

$$\therefore \boxed{H(X,Y) = H(X) + H(Y|X)} \text{ where } H(Y|X) = \sum_x \sum_y P(x,y) \log \frac{1}{P(y|x)}$$

SIMILARLY

$$\boxed{H(X,Y) = H(Y) + H(X|Y)} \text{ where } H(X|Y) = \sum_x \sum_y P(x,y) \log \frac{1}{P(x|y)}$$

DEFINE

$$\boxed{I(X;Y) = \sum_x \sum_y p(x,y) \log \frac{p(x,y)}{p(x)\,p(y)}}$$

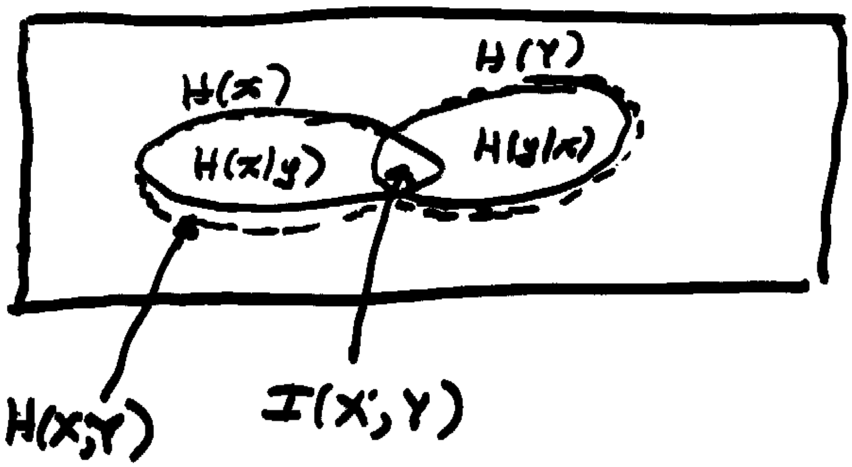
IT IS EASY TO SEE THAT

(a) $I(X,Y) = H(X) + H(Y) - H(X,Y)$

OR $H(X,Y) = H(X) + H(Y) - I(X;Y)$

(b) $I(x;y) = H(X) - H(X|Y)$

(c) $I(x;Y) = H(Y) - H(Y|X)$



A BASIC INEQUALITY

$$\boxed{I(X;Y) \geq 0} \qquad (\text{or } -I(X;Y) \leq 0)$$

with equality iff X + Y are statistically independent

Proof

$$-I_a(X;Y) = \sum_x \sum_y P[X,Y] \log_a\left[\frac{P(X)P(Y)}{P(X,Y)}\right]$$

$$= (\log_a e) \sum_x \sum_y P[X,Y] \ln \frac{P(X)P(Y)}{P(X,Y)}$$

$$\leq \log_a e \sum_x \sum_y P[X,Y] \left[\frac{P(X)P(Y)}{P(X,Y)}\right] = 0$$

equality iff $P[X,Y] = P[x]P[y]$

Q.E.D

As a consequence

$$\left.\begin{array}{l} H(X|Y) \leq H(X) \\ \\ H(Y|X) \leq H(Y) \end{array}\right\} \text{ with equality iff } X + Y \text{ are indep.}$$

Assume that a tennis match is played between two equally matched players A and B and that the first player to win 3 sets is the winner. Let X represent the outcomes of the sets. Some of the possible values of X are: AAA, BBB, ABBB, BAAA, BABAA, etc. Let Y be the number of sets played. Then Y takes on values 3, 4, or 5. Let Z represent the player who won the match. For example if X=ABBAA, them Y=5 and Z=A.

(a) Compute H(X), H(Y), and H(Z).
(b) Compute H(X,Y), H(X,Z), and H(Y,Z).
(c) Prove that H(X|Y)=H(X) - H(Y).

	X	Y		Z
$\frac{1}{8}$	AAA	3	$\frac{2}{8}=\frac{1}{4}$	A
$\frac{1}{8}$	BBB	3		B
$\frac{1}{16}$	BAAA	4	$\frac{6}{16}=\frac{3}{8}$	A
$\frac{1}{16}$	ABAA	4		A
$\frac{1}{16}$	AABA	4		A
$\frac{1}{16}$	ABBB	4		B
$\frac{1}{16}$	BABB	4		B
$\frac{1}{16}$	BBAB	4		B
$\frac{1}{32}$	BBAAA	5	$\frac{12}{32}=\frac{3}{8}$	A
$\frac{1}{32}$	BABAA	5		A
$\frac{1}{32}$	BAABA	5		A
$\frac{1}{32}$	ABBAA	5		A
$\frac{1}{32}$	ABABA	5		A
$\frac{1}{32}$	AABBA	5		A
$\frac{1}{32}$	AABBB	5		B
$\frac{1}{32}$	ABABB	5		B
$\frac{1}{32}$	ABBAB	5		B
$\frac{1}{32}$	BAABB	5		B
$\frac{1}{32}$	BABAB	5		B
$\frac{1}{32}$	BBAAB	5		B

$P_r[Z=A]=P_r[Z=B]=\frac{1}{2}$

a) $H(X) = 2\left[\frac{1}{8}\log 8\right] + 6\left[\frac{1}{16}\log 16\right] + 12\left[\frac{1}{32}\log 32\right] = \frac{6}{8} + \frac{3}{2} + \frac{15}{8} = \frac{33}{8}$ (base 2) $= 4.125$

$H(Y) = \frac{1}{4}\log 4 + 2\left(\frac{3}{8}\log\frac{8}{3}\right) = \frac{1}{2} + \frac{9}{4} - \frac{3}{4}\log 3 = \frac{11}{4} - \frac{3}{4}\log 3 = 1.56$ (base 2)

$H(Z) = \log 2 = 1$ base 2

b) $H(X,Y) = H(X) + \cancel{H(Y|X)}^{=0} = H(X) = \frac{33}{8} = 4.125$ (base 2)

$H(X,Z) = H(X) + \cancel{H(Z|X)} = H(X) = 33/8 = 4.125$ (base 2)

$H(Y,Z) = H(Y) + H(Z|Y) = H(Y) + H(Z) = \frac{15}{4} - \frac{3}{4}\log 3 = 2.56$ (base 2)

c) $H(X|Y) = \underbrace{H(X,Y)}_{H(X)+\underbrace{H(Y|X)}_{0}} - H(Y) = H(X) - H(Y)$ (since $H(Y|X)=0$) Q.E.D

$= 2.57$

$H(Y|X) = H(Z|X) = 0$

$H(Z|X) = 1$

$H(Z|X,Y) = H(Z|X) = 1$

$H(Z|Y) = H(Y,Z) - H(Y) = 2.56 - 1.56 = 1 = H(Z)$

$H(X|Z) = H(X,Z) - H(Z) = H(X) - H(Z) = 3.125$

$H(X,Y,Z) = \underbrace{H(Z|XY)}_{1} + \underbrace{H(X,Y)}_{4.125} = 5.125$ etc