ECE 154C

Spring 2009

Class Notes – Set #4

EX. 1

ONE DIE

$X_{OUTCOME} \in \{1, 2, 3, 4, 5, 6\}$

$Y \in \{odd, even\} \rightarrow \{0, 1\}$

$H(X) = \log_2 6$

$H(Y|X) = 0$

$H(X, Y) = \log_2 6$

$H(Y) = 1 = \log_2 2$

$H(X|Y) = \log_2 3$

$H(X, Y) = \log_2 6 = \log_2 6 + 0 = \log_2 2 + \log_2 3$

EXAMPLE 2)

Two dice (independent throws)

(X_1, X_2) $\quad H(X_2|X_1) = H(X_2) = \log_2 6 = 2.58496$ $\quad H(X_1, X_2) = 2\log_2 6 = \log_2 36 = 5.16993$

$Y = X_1 + X_2$

What is $H(X_1, X_2 | Y) = ?$ It is easier to calculate $H(Y|X_1, X_2)$ and $H(Y)$

$H(X_1, X_2, Y) = H(Y|X_1, X_2) + H(X_1, X_2)$

$= H(X_1, X_2 | Y) + H(Y)$

Then $\boxed{H(X_1, X_2|Y) = H(Y|X_1, X_2) + H(X_1, X_2) - H(Y)}$

$H(Y|X_1, X_2) = 0$

Thus all we need to calculate is $H(Y)$

Y	(X_1, X_2)	
2	$(1,1)$	$1/36$
3	$(1,2)$ & $(2,1)$	$2/36$
4	$(1,3), (2,2)$ & $(3,1)$	$3/36$
5		$4/36$
6		$5/36$
7	etc	$6/36$
8		$5/36$
9		$4/36$
10		$3/36$
11		$2/36$
12		$1/36$

$$H(Y) = 2 \times \left[\frac{1}{36}\log 36 + \frac{2}{36}\log_2 18 + \frac{3}{36}\log_2 12 + \frac{4}{36}\log_2 9 + \frac{5}{36}\log_2 \frac{36}{5}\right]$$

$$+ \frac{6}{36}\log_2 6$$

$$= 2\left[.14361 + .23166 + .29875 + .35221 + .39556\right]$$

$$+ .43083 = 3.27441$$

$$H(X_1, X_2 | Y) = 5.16993 - 3.27441 = 1.89552$$

Another question:

What is $H(X_1 | Y)$?

$$H(X_1, X_2 | Y) = H(X_1 | Y) + \underbrace{H(X_2 | X_1, Y)}_{H(X_2|Y)}$$

(by the fact that X_2 is indep of X_1)

But by symmetry $H(X_1 | Y) = H(X_2 | Y)$

$$\therefore H(X_1 | Y) = \frac{1}{2} H(X_1, X_2 | Y) = 0.94776$$

MUTUAL INFORMATION AND CAPACITY OF THE DISCRETE MEMORYLESS CHANNEL

DISCRETE MEMORYLESS CHANNEL

$$P_{Y_1 Y_2 \ldots Y_N | X_1 X_2 \ldots X_N}(y_1 y_2 \ldots y_N | x_1 x_2 \ldots x_N)$$

$$= P_{Y|X}(y_1|x_1)\, P_{Y|X}(y_2|x_2) \ldots P_{Y|X}(y_N|x_N)$$

- ingle input – Single Output

$P_{Y|X}(y|x)$

AN IMPORTANT PROPERTY OF A DMC IS ITS MUTUAL INFORMATION, $I(X;Y)$

BUT IN ORDER TO CALCULATE $I(X;Y)$ WE NEED TO KNOW $P_{X,Y}(x,y) = P_{Y|X}(y|x)\, \underline{\underline{P_X(x)}}$

THUS, IN ORDER TO CALCULATE $I(X;Y)$ ONE HAS TO SPECIFY AN INPUT DISTRIBUTION $P_X(x)$

The CAPACITY, C, of A DMC is The MAXIMUM $I(X;Y)$ That CAN BE Achieved over All INPUT distributions

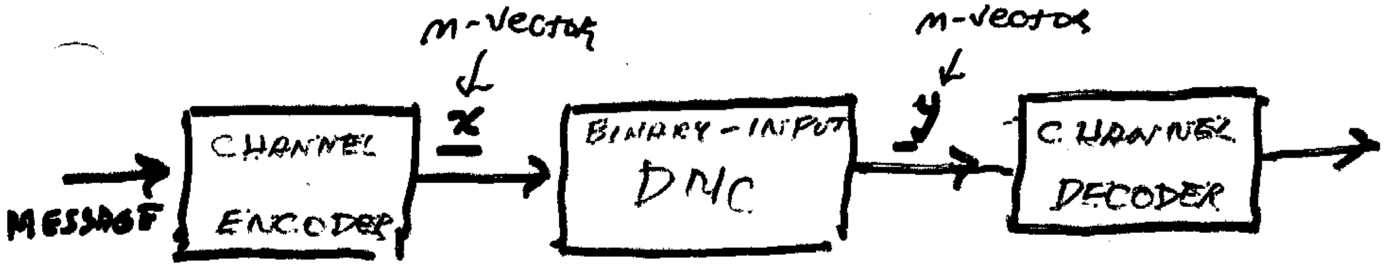
$$\boxed{C = \max_{P_X(x)} I(X;Y)}$$

ONE CAN show That for A DMC

$$\max_{P_{X_1X_2\ldots X_N}(x_1, x_2, \ldots x_N)} \frac{I(X_1X_2\ldots X_N; Y_1 Y_2 \ldots Y_N)}{N} =$$

$$\max_{P_X(x)} I(X;Y) = C$$

CODING FOR A BINARY INPUT DMC



BLOCK CODE (BINARY)

A collection of 2^{mR} binary vectors of length m.

MESSAGE: A NUMBER BETWEEN 1 and 2^{mR}. Equivalently it could be a binary vector of length mR.

ASSUMPTION: Messages occur with equal probabilities

$$Pr[\text{message} = i] = \frac{1}{2^{mR}} \quad i = 1, 2, \ldots, 2^{mR}$$

Channel Decoder: Chooses most likely code word (or equivalently most likely message) based upon received vector $\underline{y}$.

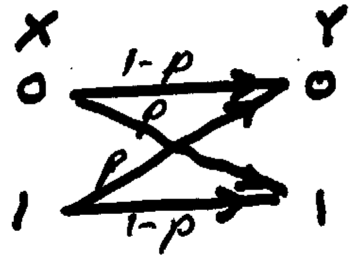
Channel Coding Theorem: For m large enough and $R < C$, (base 2) there exists an encoder and a decoder such that

Pr [message produced by decoder $\neq$ original message] $< \varepsilon$

for any $\varepsilon > 0$.

Computation of Capacity

imple   B.S.C.



INPUT X \ OUTPUT Y	0	1
0	(1-p)	p
1	p	(1-p)

↑ $P_{Y|X}(y|x)$

$$I_2(X;Y) = H_2(Y) - H_2(Y|X)$$

$$H_2(Y|X) = \sum_{x=0}^{1}\sum_{y=0}^{1} P_X(x)\, P_{Y|X}(y|x) \log_2 \frac{1}{P_{Y|X}(y|x)}$$

$$= \sum_{x=0}^{1} P_X(x)\left((1-p)\log_2\frac{1}{(1-p)} + p\log_2\frac{1}{p}\right)$$

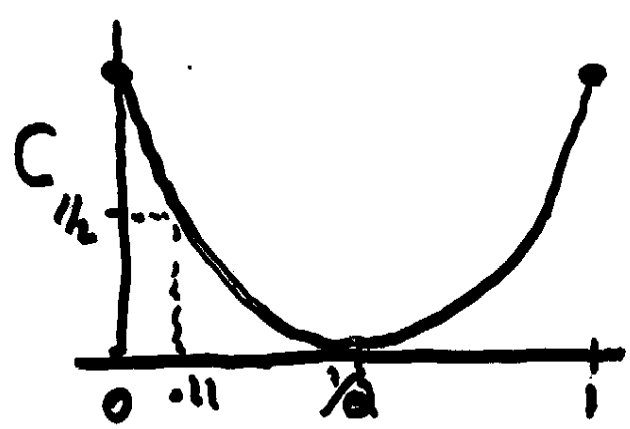
Define $\quad h_2(p) = (1-p)\log_2\frac{1}{(1-p)} + p\log_2\frac{1}{p}$

$$\therefore\; H_2(Y|X) = h_2(p)$$

But $H_2(Y) \le \log_2 2$ with equality iff $P[Y=0]=P[Y=1]=1/2$

But if $P_X(0) = 1/2 = P_X(1) = 1/2$, then $P_Y(0) = P_Y(1) = 1/2$

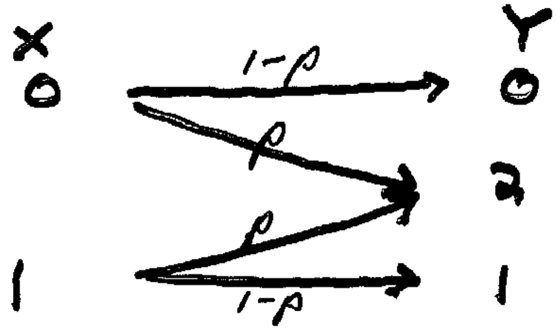
$$\therefore\; \boxed{C = \max_{P_X(x)} I(X;Y) = \log_2 2 - h_2(p) = 1 - h_2(p)}$$



COMPUTATION OF CAPACITY

EXAMPLE

BINARY ERASURE CHANNEL (BEC)



INPUT \ OUTPUT	0	1	2
0	1-p	0	p
1	0	1-p	p

ASSUME $P[X=0]=\alpha$ $\quad P[X=1]=(1-\alpha)$

Then $H_2(X)=\alpha\log_2\frac{1}{\alpha}+(1-\alpha)\log_2\frac{1}{(1-\alpha)}=h_2(\alpha)$

BUT $\quad I_2(X;Y)=H_2(X)-H(X|Y)$

$$H_2(X|Y)=\sum_{y=0}^{2}P_Y(y)\underbrace{\sum_{x=0}^{1}P_{X|Y}(x|y)\log_2\frac{1}{P_{X|Y}(x|y)}}_{H(X|Y=y)}$$

FOR THAT CHOICE OF $P_X(x)$, then $P_Y(0)=\alpha(1-p)$,

$P_Y[1]=(1-\alpha)(1-p)$, $P_Y(2)=p$.

ALSO

$H(X|Y=0)\sum_{x=0}^{1}P_{X|Y}(x|0)\log_2\frac{1}{P_{X|Y}(x|0)}=0$

$H(X|Y=1)\sum_{x=0}^{1}P_{X|Y}(x|1)\log_2\frac{1}{P_{X|Y}(x|1)}=0$

$H(X|Y=2)\sum_{x=0}^{1}P_{X|Y}(x|2)\log_2\frac{1}{P_{X|Y}(x|2)}=\alpha\log_2\left(\frac{1}{\alpha}\right)+(1-\alpha)\log_2\frac{1}{(1-\alpha)}=h_2(\alpha)$

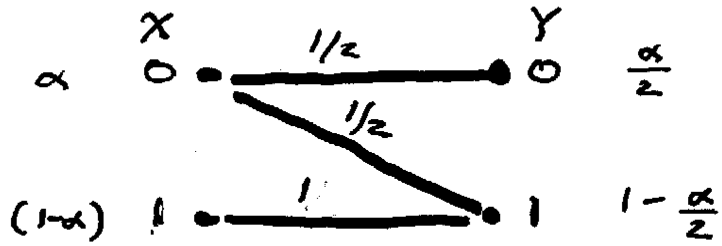
$H(X|Y)=p\,h_2(\alpha)$

thus $C=\max_\alpha\left[H(X)-H(X|Y)\right]=\max_\alpha\left[h_2(\alpha)-p\,h_2(\alpha)\right]=\max_\alpha h_2(\alpha)(1-p$

$\boxed{C=1-p}$

COMPUTATION OF CAPACITY

EXAMPLE



IN X \ OUT Y	0	1
0	1/2	1/2
1	0	1

$P_{Y|X}(y|x)$

$$I_2(X;Y) = H_2(Y) - H_2(Y|X) = h_2\left(\frac{\alpha}{2}\right) - \alpha h_2(1/2)$$

$$= \frac{\alpha}{2}\log_2\frac{2}{\alpha} + \left(1-\frac{\alpha}{2}\right)\log_2\frac{1}{\left(1-\frac{\alpha}{2}\right)} - \alpha\, h_2(1/2)$$

$$\therefore I_e(X;Y) = \frac{\alpha}{2}\ln\frac{2}{\alpha} + \left(1-\frac{\alpha}{2}\right)\ln\frac{1}{\left(1-\frac{\alpha}{2}\right)} - \alpha\, h_e(1/2)$$

We will maximize $I_e(X;Y)$ with respect to α. This same α will also maximize $I_2(X;Y)$

$$0 = \frac{\partial I_e(X;Y)}{\partial\alpha} = \frac{1}{2}\ln\frac{2}{\alpha} - \frac{\alpha}{2}\cdot\frac{1}{\alpha} + \left(1-\frac{\alpha}{2}\right)\frac{1/2}{(1-\alpha/2)} - \frac{1}{2}\ln\frac{1}{\left(1-\frac{\alpha}{2}\right)}$$

$$- h_e\left(\tfrac{1}{2}\right) \leftarrow \ln 2$$

$$0 = \frac{1}{2}\ln 2 - \frac{1}{2}\ln\alpha - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} + \frac{1}{2}\ln\left(1-\frac{\alpha}{2}\right) - \ln 2$$

$$\frac{1}{2}\ln\frac{\alpha}{1-\frac{\alpha}{2}} = -\frac{1}{2}\ln 2 = \frac{1}{2}\ln\frac{1}{2}$$

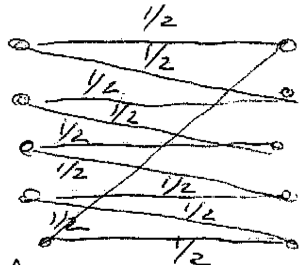
$$\therefore \frac{1-\frac{\alpha}{2}}{\alpha} = 2 \Rightarrow 1-\frac{\alpha}{2} = 2\alpha \Rightarrow 1 = \frac{5}{2}\alpha \Rightarrow \boxed{\alpha = 2/5}$$

$$C_2 = I_2(X;Y)\Big|_{\alpha=2/5} = \frac{1}{5}\log_2 5 + \frac{4}{5}\log_2\frac{5}{4} - \frac{2}{5}$$

$$\boxed{C_2 = \log_2 5 - 2 = .3219}$$

EXAMPLE of Coding

Channel



Code Words		OUTPUTS			
00	$\rightarrow$	00	01	10	11
12	$\rightarrow$	12	12	22	23
24	$\rightarrow$	24	20	34	30
31	$\rightarrow$	31	32	41	42
43	$\rightarrow$	43	44	03	04

Decoder Works in reverse.

There are 5 code words. Thus each code word can represent one 5-ary digit or $\log_2 5$ binary digits.

Thus since each code word represents 2 uses of the channel the rate of the code is

$1/2 = 0.5$ 5-ary digit per channel use

or

$\frac{1}{2} \log_2 5 = 1.1609$ binary digit per channel use

$$C = \max_{P(X)} I(X;Y) = I(X;Y)\Big|_{P[X=x]=1/5 \quad x=0,1,2,3,4}$$

$$I(X;Y) = H(Y) - H(Y/X)$$

If inputs are equally likely, outputs are equally likely. Thus

$H(Y) = \log 5$

But for any value of the input, there are two possible outputs that are equally likely

Thus $H(Y|X) = \log 2$

$$C = \log 5 - \log 2 = \log 5/2$$

Base 5 $\quad C_5 = \log_5 \frac{5}{2} = 0.569 \qquad$ Note $\quad 0.5 < 0.569$

Base 2 $\quad C_2 = \log_2 \frac{5}{2} = 1.322. \qquad$ Note $\quad 1.16 < 1.322$

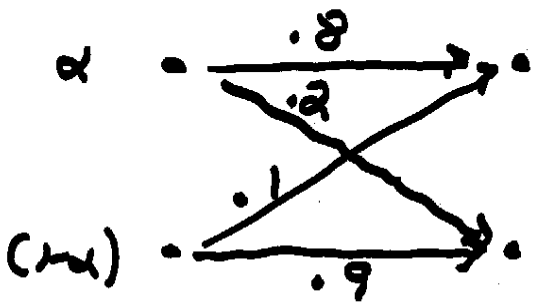
FINDING THE CAPACITY USING A COMPUTER

FOR AN ARBITRARY DMC, ONE CANNOT, IN GENERAL, USE ANALYTIC TECHNIQUES TO FIND THE CHANNEL CAPACITY.

INSTEAD ONE CAN USE A COMPUTER TO SEARCH OUT THE INPUT DISTRIBUTION, $P_X(x)$, THAT MAXIMIZES $I(X;Y)$

THERE IS A SPECIAL ALGORITHM, CALLED THE BLAHUT-ARIMOTO ALGORITHM, FOR DOING THIS. FOR SIMPLE CASES, A BRUTE-FORCE SEARCH CAN BE USED.

EXAMPLE



- USE COMPUTER TO EVALUATE $I(X;Y) = f(\alpha)$
- OPTIMIZE WITH RESPECT TO α.