UCSD ECE 154C
Communications Systems III
Prof. Jack Keil Wolf

Solutions to ECE 154C Problem Set 6

1. **Solution:**

   We can prove this by using the fact that all codewords are linear combinations of each other. Let $M = 2^k$ be the total number of codewords.

   First, all zero column is possible since the sum of any two codewords with zero in that column will also have zero in that column.

   Now let's consider the other case.Let a column have α 1's in the column and $(M - \alpha)$ 0's. Choose one codeword with 1 and add it to all words that have 1 in that column. Then, $M - \alpha \geq \alpha$ since there could be at least α codewords with 0 in that column.

   Now add that codeword to all the words having 0 in that column. This produces $M - \alpha$ words with 1 in that column. Likewise, $\alpha \geq M - \alpha$. From the two inequalities,

$$\alpha = M - \alpha$$
$$\alpha = \frac{M}{2}$$

2. **Solution:** (a)

$$\tilde{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

   (b) $d_{min} = 4$ since 4 columns of the $\tilde{H}$ matrix sum to $\bar{0}$. Therefore,

$$t = \lfloor \frac{d_{min} - 1}{2} \rfloor = 1$$

   (c) The code fails to detect error patterns that match non-zero codewords. By observing the generator matrix, we can see that the all 1 codeword is a codeword. From problem 1, we see that the number of 0's and 1's are the same if we arrange the codewords as rows of an array. Hence, we can only get codes with Hamming weights 0,4 and 8.

Therefore we will have 1 word with $w_H = 0$, 14 words with $w_H = 4$ and 1 word with $w_H = 8$, where w_H denotes the Hamming weight. Hence the code will fail to detect 15 error patterns. We can find the following probability.

$$Pr(\text{undetected error}) = 14p^4(1-p)^4 + p^8$$

where p is the bit error probability.

(d)
The code is guaranteed to correct up to 3 erasures. There are $\binom{8}{1} = 8$ patterns with 1 erasure, $\binom{8}{2} = 28$ patterns with 2 erasures and $\binom{8}{3} = 56$ patterns with 3 erasures.

For erasure patterns with 4 erasures, we can select 3 out of 8 places for the erasure. However, the 4th place depends on the previous 3 selections. Hence we can correct $\binom{8}{3} = 56$ patterns with 4 erasures. We cannot correct any pattern with more than 4 erasures. We can find the following probability.

$$Pr(\text{uncorrected codeword}) = 1 - [8p(1-p)^7 + 28p^2(1-p)^6 + 56p^3(1-p)^5 + 56p^4(1-p)^4]$$

3. **Solution:**
The table used to compute the parity digits from the information digits at the encoder can be given as the following.

info bits	parity bits
0000	0000
0001	1110
0010	1101
0011	0011
0100	1011
0101	0101
0110	0110
0111	1000
1000	0111
1001	1001
1010	1010
1011	0100
1100	1100
1101	0010
1110	0001
1111	1111

Let $\bar{R} = [r_1, ..., r_8]$ be the received symbol. We can find the syndrome by doing the following. First using $[r_1, ..., r_4]$ as information bits and finding the parity bits $[p_1, ..., p_4]$ corresponding to the 4 bits. Then

$$\bar{S} = [r_5, ..., r_8] \oplus [p_1, ..., p_4]$$

where $\bar{S}$ is the syndrome of the received symbol.

4. **Solution:**
The parity matrix of the $(15, 11)$ Hamming single error correcting code is

$$\tilde{H} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a)We can insert a extra parity digit by first adding a all zero column at the end of the $\tilde{H}$ matrix and then adding an all one row to the same matrix. Then,

$$\tilde{H} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

In order to make the last 5 columns of the matrix to be the unit matrix, we add the first 4 rows to the last row which gives,

$$\tilde{H} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The generator matrix can be acquired by using the fact that

$$\tilde{G} = \begin{bmatrix} I_{11\times 11} & A \end{bmatrix}$$

5. **Solution:**
By considering the last row and last column as the parity digits of the rows and columns, the problem is now to find the number of (6×3) distinct matrices with binary inputs. Since the number of binary sequences of length 3 is 8, we have to consider 8 different combinations for each row. Then the answer is simply $8 \times 8 \times 8 \times 8 \times 8 \times 8 = 262144$.