Equivalence of Formulas for Calculations at Parity Nodes in LDPC Decoding

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Let us consider a parity node where L code bits must sum to 0, modulo 2 . Assume that P_i is the probability that the i-th code bit in this parity equation is equal to 0, for i=1, 2, …, L.

In LDPC decoding, the parity node makes new estimates for these probabilities based upon the fact that there must be an <u>even number of 1's</u> in these L bits. Call these new estimates of the probabilities P'_i. The new probability that bit i is equal to 0 is then equal to the probability that there are an <u>even number of 1's in the other L-1 bits</u>. This is an ugly combinatoric formula if L is large so we seek simpler expressions.

In what follows we give three simpler expressions for the new probabilities. For ease of notation we will phrase our result in terms of the new probability that bit node L is equal to 0, namely P'_L. We first define two new quantities:

$Q_i=1-P_i$ (the probability that bit node i is equal to 1)

and

$\Lambda_i=\ln[P_i/(1-P_i)]$ (the log-likelihood ratio for bit node i).

Then the three simpler expressions are:

$$(P'_L-Q'_L) = (P_1-Q_1)(P_2-Q_2)\ldots(P_{L-1}-Q_{L-1}), \tag{1}$$

$$(2P'_L-1) = (2P_1-1)(2P_2-1)\ldots(2P_{L-1}-1), \tag{2}$$

$$\tanh(\Lambda'_L/2) = \tanh(\Lambda_1/2)\tanh(\Lambda_2/2)\ldots\tanh(\Lambda_{L-1}/2). \tag{3}$$

It is the last of these expressions that is most often seen in the literature. The second expression is particularly interesting since it is easy to understand and may lead to some intuition as to how to use quantization.

We first show the equivalence of these three expressions. The equivalence between (1) and (2) is obvious since (2) follows from (1) by substituting $Q_i=1-P_i$. We now show the equivalence of (1) and (3). From the definition of the log-likelihood ratio,

$$\Lambda=\ln[P/(1-P)].$$

Then

$$P/(1-P) = e^{\Lambda}$$

and solving for P yields

$$P= e^{\Lambda}/(e^{\Lambda} +1).$$

But then

$$\begin{aligned} 2P - 1 &= (2 e^{\Lambda}/(e^{\Lambda} +1)) - 1 \\ &= (2e^{\Lambda}-e^{\Lambda}-1)/(e^{\Lambda} +1) \\ &= (e^{\Lambda}-1)/(e^{\Lambda} +1) \\ &= (e^{\Lambda/2} - e^{-\Lambda/2})/(e^{\Lambda/2} + e^{-\Lambda/2}) \\ &= \tanh(\Lambda/2) \end{aligned}$$

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Finally, we give a derivation for (1) based upon a lecture by Andrew Viterbi at UCSD.

Let $P^{(L-2)}$ be the probability of an even number of 1's in bits 1, 2, …, L-2 and let $Q^{(L-2)} = 1 - P^{(L-2)}$ equal the probability that there are an odd number of 1's in those bits. Then:

$$P^{(L-1)} = P^{(L-2)} P_{L-1} + Q^{(L-2)} Q_{L-1}.$$

and

$$Q^{(L-1)} = P^{(L-2)} Q_{L-1} + Q^{(L-2)} P_{L-1}.$$

Then

$$(P^{(L-1)} - Q^{(L-1)}) = (P^{(L-2)} - Q^{(L-2)})(P_{L-1}- Q_{L-1}).$$

Clearly (1) holds for L=2 . It is also easy to show that it holds for L=3. Now assume it holds for L replaced by L-2. From the above equation we see that it holds for L replaced by L-1. This completes the proof.

Finally note that the mapping Z= (2P-1) maps the bit probability P=0 into Z= -1 and the bit probability P=1 into Z=+1 with a straight line connecting these points. But P=0 means that the bit is a binary 1 and P=1 means that the bit is a binary 0. Thus a binary 1 is mapped to -1 and a binary 0 is mapped to +1. This is the same mapping that makes modulo 2 addition isomorphic to multiplication of +1's and -1's. The linear interpolation between these points extends the concept of modulo 2 addition to soft decisions.

Formula (2) is also informative with respect to choosing quantization regions for the metrics.