Lecture 2

2.1 Laser model based upon an Electric Dipole oscillator

Last week we derived the net induced transition. When N_1 is larger than N_2, we have the net absorption of photons instead of the amplification. However, the absorption (or attenuation) of materials is related to the imaginary part of the complex susceptibility χ. After some algebra, we can obtain:

$$\chi''(\nu) = (N_2 - N_1)\,\frac{c^3}{8\pi n^3 h\nu^3 t_{spont}}\,g(\nu) \qquad (1)$$

where g(ν) is the lineshape function related to the transition (one can relate g(ν) to the uncertainty of energy of either level 1 or 2).

One can use a simple electric-dipole oscillator model for understanding the atomic transition related to Eq. 1. Assuming the electron oscillates around a nucleus like an electric dipole with a dipole moment $\mu(t)$. Since the mass of electron is much smaller than that of the nucleus, the nucleus can be assumed stationary. The bonding between the electron and the nucleus is modeled as an elastic spring with a spring constant k. The frictional force is assumed present to damp out the oscillation. As usual, the equation of motion for the electron can be described as:

$$\frac{d^2x(t)}{dt^2} + \sigma\frac{dx(t)}{dt} + \frac{k}{m}x(t) = -\frac{e}{m}E(t) \qquad (2)$$

where E(t) is the electric field. Let's assume that both E and x are real part of the corresponding complex quantities, that is:

$$E(t) = \mathrm{Re}\left(E\,e^{i\omega t}\right)$$
$$x(t) = \mathrm{Re}\left(x(\omega)\,e^{i\omega t}\right)$$

Let's define the resonant frequency $\omega_o^2 = k/m$, substituting the above expressions into Eq. 2, we obtain,

$$(\omega_o^2 - \omega^2)\,x(\omega) + i\omega\sigma\,x(\omega) = -\frac{e}{m}E \qquad (3)$$

or,

$$x(\omega) = -\frac{\frac{e}{m} E}{\omega_o^{\,2} - \omega^2 + i\omega\sigma} \tag{4}$$

and for frequency close to the resonant frequency, the denominator can be simplified to,

$$x(\omega) = -\frac{\frac{e}{m} E}{2\omega_o(\omega_o - \omega) + i\omega_o\sigma} \tag{5}$$

Next we need to relate displacement of electron (from equilibrium position), x, to the dipole moment μ and the medium polarization P,

$$\mu(t) = -\, e\, x(t)$$
$$P(t) = \mathrm{Re}\left(P(\omega)\, e^{i\omega t}\right)$$

$P(\omega)$ is related to $x(\omega)$,

$$\begin{aligned} P(\omega) = -\,Ne\, x(\omega) &= \frac{N\frac{e}{m} E}{2\omega_o(\omega_o - \omega) + i\omega_o\sigma} \\ &= \frac{-\,iN\dfrac{e^2}{m\omega_o\sigma}}{1 + i\dfrac{2(\omega - \omega_o)}{\sigma}}\, E \end{aligned} \tag{6}$$

and furthermore, P is also defined in term of the susceptibility χ through

$$P_{transition}(\omega) = \varepsilon_o\, \chi(\omega) E \tag{7}$$

Therefore, from Eqs. 6, 7 we get,

$$\chi(\omega) = -\,iN\frac{e^2}{m\omega_o\sigma\, \varepsilon_o}\, \frac{1 - i\dfrac{2(\omega - \omega_o)}{\sigma}}{1 + \dfrac{4(\omega - \omega_o)^2}{\sigma^2}} \tag{8}$$

Define the real and imaginary part of χ,

$$\chi(\omega) = \chi'(\omega) - i\chi''(\omega) \tag{9}$$

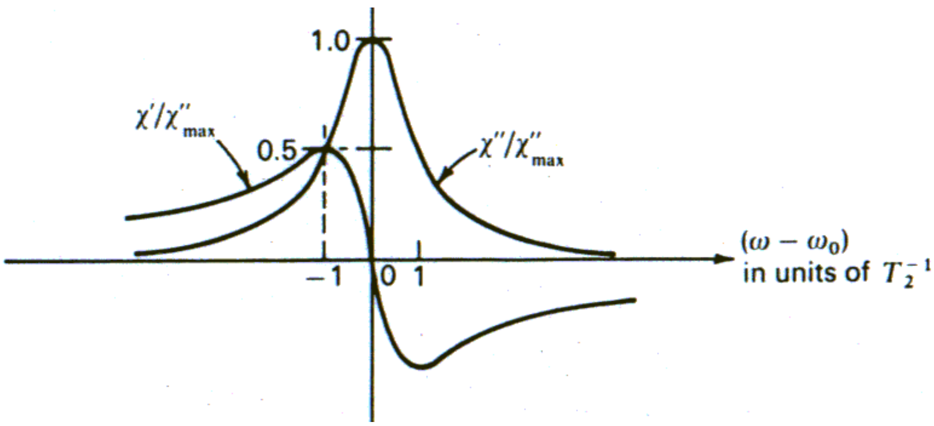
we have,

$$\chi'(\omega) = N\frac{e^2}{m\omega_o\sigma\,\varepsilon_o}\,\frac{\frac{2(\omega-\omega_o)}{\sigma}}{1+\frac{4(\omega-\omega_o)^2}{\sigma^2}} \qquad (10)$$

and

$$\chi''(\omega) = -i\,N\frac{e^2}{m\omega_o\sigma\,\varepsilon_o}\,\frac{1}{1+\frac{4(\omega-\omega_o)^2}{\sigma^2}}, \qquad (11)$$

which can be depicted as follows,



From Maxwell's equations, the displacement vector **D** is related to the polarization by,

$$D = \varepsilon_o E + P + P_{transition} = \varepsilon E + \varepsilon_o\,\chi(\omega)E$$
$$= \varepsilon\left(1+\frac{\varepsilon_o}{\varepsilon}\chi(\omega)\right)E = \varepsilon'(\omega)\,E \qquad (12)$$

Here we define a dielectric permitivity function ε'(ω) which is related to the lasing transition. For the electromagnetic wave, the electric field can be expressed in terms of ,

$$e(z,t) = \mathrm{Re}\left(E\,e^{i(\omega t - k'z)}\right)$$

In this form, the modified propagation constant k' is related to the new permitivity through,

$$k'(z) = \omega\sqrt{\mu\varepsilon'} \approx k\left(1+\frac{\varepsilon_o}{2\varepsilon}\chi\right) = k\left(1+\frac{\chi'}{2n^2}\right) - i\frac{k\chi''}{2n^2} \qquad (13)$$

The first factor represents the phase change, while the second factor represents the amplitude change via amplification or attenuation. Usually we define the phase change as Δk, attenuation as γ,

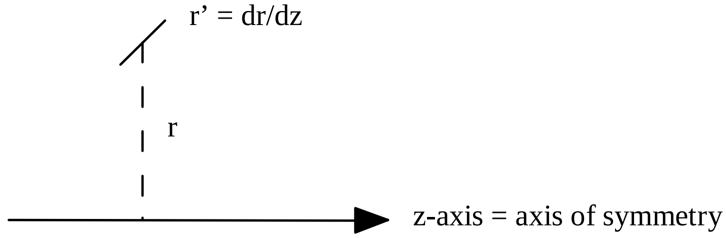
$$\Delta k = k\frac{\chi'(\omega)}{2n^2}$$
$$\gamma = -k\frac{\chi''(\omega)}{n^2} \qquad (14)$$

And the electromagnetic wave is then given by,

$$e(z,t) = \mathrm{Re}\left(E\, e^{i(\omega t\ ?\ k+\Delta k)z)} e^{\frac{\gamma}{2}z}\right) \qquad (15)$$

2.2. <u>Cavity Stability </u>(this part is a review)

We first review the cavity stability condition using the ray tracing method.
In the paraxial approximation, the ray is denoted by both the displacement r from an (optical) axis of symmetry and the slope of change of r with respect to the propagation along the axis:



We use a 2 x 1 column vector for each point of the ray, and a 2 x2 matrix to designate any optical element that the ray comes across.

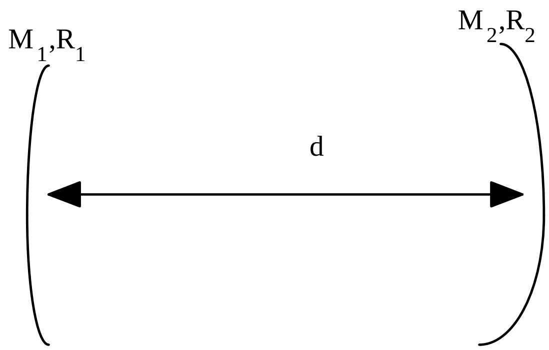
$$\begin{pmatrix} r \\ r' \end{pmatrix} = \begin{pmatrix} \text{vertical position} \\ \text{slope w.r.t z} \end{pmatrix}$$

Example of optical elements:

$$\text{Lens: } \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}, \qquad \text{distance d : } \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

Mirror of R: $\begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix}$, etc.

An optical cavity is stable if the ray position stays close the optical axis after many transits between the mirrors (in this case it is bounded by the mirrors), otherwise it is unstable. If the mirrors must be perfectly aligned in order to keep the ray near the axis, it is conditionally stable.



For a cavity with two mirrors (with radius of curvature R1, R2 respectively) separated by a distance d, it can be shown that the transfer matrix T after one round trip takes the form:

$$T = \begin{pmatrix} 1-\frac{d}{f_2} & d + d\left(1-\frac{d}{f_2}\right) \\ -\frac{1}{f_1} - \frac{1}{f_2}\left(1-\frac{d}{f_1}\right) & \left(1-\frac{d}{f_1}\right)\left(1-\frac{d}{f_2}\right) - \frac{d}{f_1} \end{pmatrix}$$

Let's use the ABCD notation for T, and let s, s+1 denote the s^{th} and $s+1^{th}$ round trip,

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (16)$$

Therefore,

$$\begin{pmatrix} r_{s+1} \\ r'_{s+1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} r_s \\ r'_s \end{pmatrix} \quad (17)$$

or,

$$r_{s+1} = Ar_s + Br'_s \quad (18)$$
$$r'_{s+1} = Cr_s + Dr'_s \quad (19)$$

Solve for r'_s from Eq. 18,

$$r'_s = \frac{1}{B}(r_{s+1} - Ar_s) \qquad (20)$$

Permuting s to s+1,

$$r'_{s+1} = \frac{1}{B}(r_{s+2} - Ar_{s+1}) = Cr_s + Dr'_s$$

$$\text{(using Eq. 20)} \qquad = Cr_s + \frac{D}{B}(r_{s+1} - Ar_s) \qquad (21)$$

This gives, after some arrangement, (and using AD-BC=1)

$$r_{s+2} - 2\left(\frac{A+D}{2}\right) r_{s+1} + r_s = 0 \qquad (22)$$

Assuming r oscillates in phase and maintains a fixed amplitude over many round trips,

$$r_s = r_o e^{is\theta}$$

Inserting this into Eq. 22,

$$e^{i\theta} = \left(\frac{A+D}{2}\right) \pm i\sqrt{\left(1 - \left(\frac{A+D}{2}\right)^2\right)} \qquad (23)$$

To ensure the stability condition, therefore, θ must be real, which implies the left hand side of Eq. 23 is equal to cos(θ) ± i sin(θ), for which

$$-1 \le \frac{A+D}{2} \le 1, \text{ or }, \quad 0 \le \frac{A+D+2}{4} \le 1 \qquad (24)$$

Note, after some algebra,

$$\frac{A+D+2}{4} = \left(1 - \frac{d}{2f_1}\right)\left(1 - \frac{d}{2f_2}\right)$$

Using Eq. 24, we finally arrive at the cavity stability condition,

$$0 \le \left(1 - \frac{d}{2f_1}\right)\left(1 - \frac{d}{2f_2}\right) \le 1 \qquad (25)$$

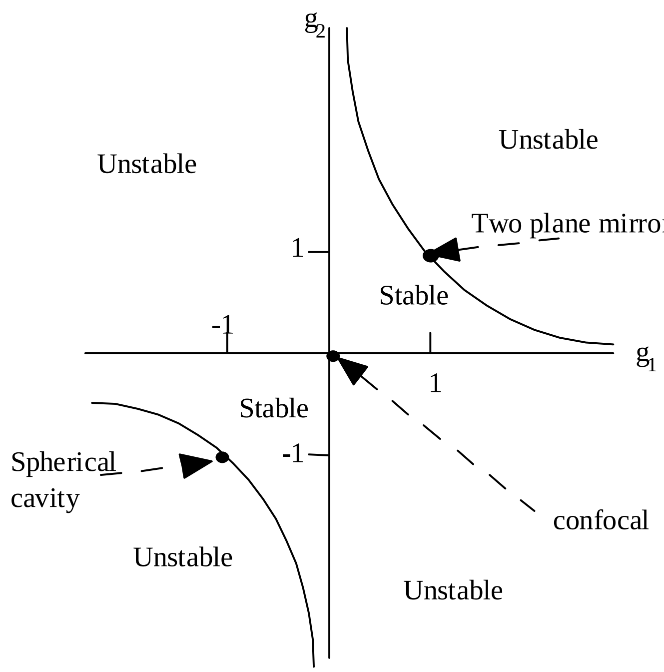
2.2 Cavity Stability (continued)

Eq. 25 is plotted below with the following notations for the axes:

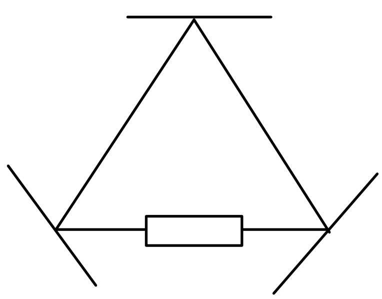
$$g_1 = \left(1 - \frac{d}{2f_1}\right)$$
$$g_2 = \left(1 - \frac{d}{2f_2}\right)$$

With these Eq. 25 reads:

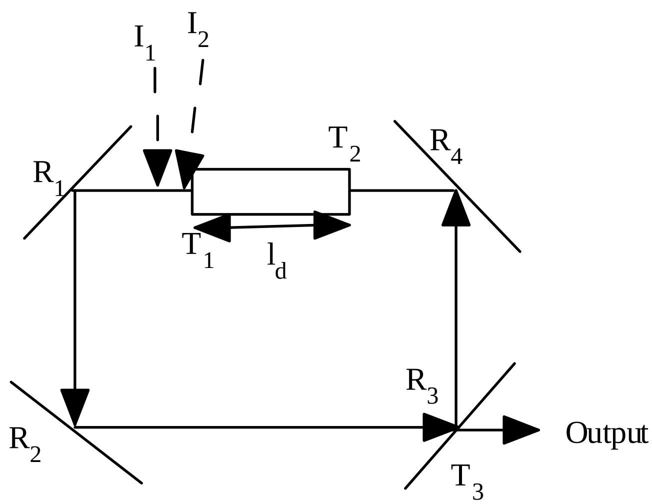
$$0 \le g_1 g_2 \le 1 \qquad (1)$$



Note that the planar mirror cavity, the confocal cavity, and the spherical cavity are conditional stable points in the plot as they border the stable and unstable regions. On the other hand, unstable cavity configuration can be useful for high power operation. There are other common designs for lasers. A particular one is the ring laser cavity that consists of several mirrors in a folded configuration. For instance,



We can apply the transfer matrix over a round trip and apply the stability condition as before for examining this configuration. As another example, consider the four-mirror cavity below:



The intensity of the light beam immediately after the laser medium can be written as:

$$I_2 = R_1R_2R_3R_4T_2e^{-\alpha_d l_d}T_1I_1 \qquad (2)$$

where α_d denote the loss of the medium. For the case of small gain, we can just replace α_d by the gain coefficient γ_d with a sign change. However, when the intensity is very large we have to consider the entire intensity evolution from one end to the other, including the effect of gain saturation. For the sake of simplicity, let's consider the case of homogeneously broadened medium and define I_s as the saturated intensity and γ_o as the unsaturated gain:

$$\frac{dI}{dz} = \frac{\gamma_o I}{1 + \frac{I}{I_s}} \qquad (3)$$

Only in the limit of small I can the intensity be approximated by an exponential function of z.

2.3. Gaussian Beam Approximation

In the reference text – Quantum Electronics, the Gaussian beam approximation is discussed in detail. The transverse electric field profile is given as:

$$G(x,y) = e^{-\frac{x^2}{\omega_o^2}} e^{-\frac{y^2}{\omega_o^2}} \frac{\omega_o}{\omega(z)} \exp\left(-j\left(kz - \tan^{-1}\left(\frac{z}{z_o}\right)\right)\right) \exp\left(-j\frac{kr^2}{2R(z)}\right) \quad (4)$$

where $r^2 = x^2 + y^2$. The beam size function, ω, and the radius of curvature function, R, are given as:

$$\omega^2(z) = \omega_o^2\left(1 + \left(\frac{\lambda_o z}{\pi n \omega_0^2}\right)^2\right) \quad (5)$$

$$R(z) = z\left(1 + \left(\frac{\pi n \omega_0^2}{\lambda_o z}\right)^2\right) \quad (6)$$

ω_o is the minimum spot size. The first three factors in Eq. 4 affect the amplitude, the next factor affects the phase along the z-direction, the last factor affects the transverse phase in the x,y plane.

Case 1: For very large R, by inspection, we can ignore the factor of 1 in the bracket of Eq. 6. Now for z = d/2, (i.e. taking the origin of the z-coordinate at one of the mirrors), we get,

$$R \approx \frac{2\pi^2 n^2 \omega_0^4}{\lambda_o^2 d} \quad (7)$$

thereby,

$$\omega_o^4 \approx \left(\frac{\lambda}{\pi}\right)^2\left(R\frac{d}{2}\right) \quad (8)$$

Case 2: Confocal, R = d, at the center, z = d/2, we have, using Eq. 6,

$$R = \frac{d}{2}\left(1 + \left(\frac{2\pi n\omega_0^{\ 2}}{\lambda_o d}\right)^2\right) = d$$

or, after some algebra,

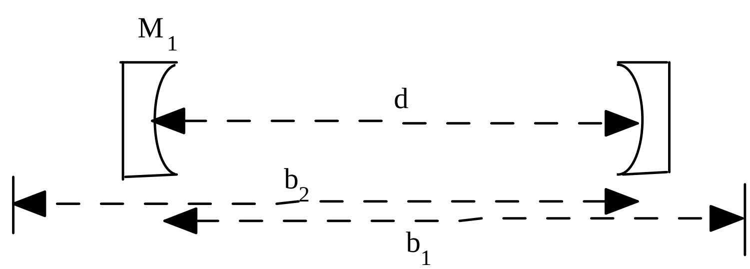
$$\omega_o = \sqrt{\frac{d\lambda}{\pi}} = \sqrt{\frac{R\lambda}{\pi}} \qquad (9)$$

The ω_o in Eq. 9 corresponds to the smallest laser tube size for a given d.

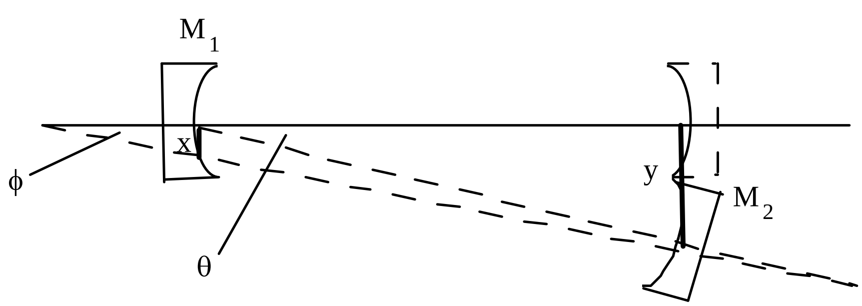
We can apply Eqs. 5 and 6 in the analysis of other cavity configurations.

2.4. Alignment Sensitivity of Laser

In the He-Ne laser experiment this week you will study further the alignment sensitivity of laser cavity. Let's suppose the mirrors have a radius of curvature b_1 and b_2 respectively, and are at a distance d apart,



If M_2 is angularly misaligned by an angle θ with respect to M_1, using the notations in the following figure, we see that,



Since the arc can be described in term of θ or ϕ, we have (after considering the intersection of the two lines):

$$b_1\theta = (b_1 + (b_2 - d))\phi \qquad (10)$$

Defining x, as the transverse displacement of the beam spot at M_1, and y as transverse displacement of the beam spot at M_2, as M_2 is angularly misaligned, we have thus,

$$x = (b_2 - d)\phi = \frac{b_1\theta(b_2 - d)}{(b_2 + b_1 - d)} \qquad (11)$$

$$y = b_2\phi = \frac{b_1 b_2\theta}{(b_2 + b_1 - d)} \qquad (12)$$

The sensitivity of alignment (S) is measured with respect to spot size ω as,

$$S_x^{-1} = \omega_x \frac{\theta}{x} = \omega_x \frac{(b_2 + b_1 - d)}{b_1 (b_2 - d)} \qquad (13)$$

$$S_y^{-1} = \omega_y \frac{\theta}{y} = \omega_y \frac{(b_2 + b_1 - d)}{b_1 b_2} \qquad (14)$$

We can use Eqs. 13 and 14 for examining the alignment sensitivity of various types of cavity.

(A) Plane mirror cavity, b_1 and b_2 approach infinity, this implies $S^{-1} = 0$ or it is very sensitive to misalignment.

(B) For large radius of curvature case, substitute R = b into Eq. 8,

$$\omega_o^4 \approx \left(\frac{\lambda}{\pi}\right)\left(b\frac{d}{2}\right)$$

for b >> d, Eq. 14 gives,

$$S^{-1} \approx 2\frac{\omega}{b}$$

which is not as critical as case A.

(C) For confocal cavity, $b_1 = b_2 = d$, Eq. 11 implies x = 0, then

$$S_x^{-1} \rightarrow \infty$$

On the other hand, $y = b\theta$, so

$$S_y^{-1} \rightarrow \frac{\omega_y}{b}$$

and is therefore the least sensitive so far.

Appendix

TABLE 10.6 Data Associated with the Various States of Neon

Transition	J_{upper}	J_{lower}	λ (Å)	$A(10^6\ sec^{-1})$	Relative Intensity
$3s_2 \rightarrow 2p_1$	1	0	7304.9	0.48	30
$3s_2 \rightarrow 2p_2$	1	1	6401.1	0.6 (est.)	100
$3s_2 \rightarrow 2p_3$	1	0	6351.9	0.7	100 (common "red" laser)
$3s_2 \rightarrow 2p_4$	1	2	6328.2	6.56	300 (common "red" laser)
$3s_2 \rightarrow 2p_5$	1	1	6293.8	1.35	100 (common "red" laser)
$3s_2 \rightarrow 2p_6$	1	2	6118.0	1.28	100
$3s_2 \rightarrow 2p_7$	1	1	6046.1	0.68	50
$3s_2 \rightarrow 2p_8$	1	2	5939.3	0.56	50
$3s_2 \rightarrow 2p_9$	1	3	5882.5	Forbid $\Delta J = 2$	Not observed
$3s_2 \rightarrow 2p_{10}$	1	1	5433.6	0.59	250
$3s_2 \rightarrow \Sigma 2p$	1	—	Red-orange	12.8	—
$3s_2 \rightarrow 3p_4$	1	2	33913	2.87	—
$3s_2 \rightarrow \Sigma 3p$	1	—	IR	5.24	—
$2p_4 \rightarrow 1s_2$	2	1	6678.3	23.8	500
$2p_4 \rightarrow 1s_3$	2	0	6234.5	Forbid $\Delta J = 2$	Not observed
$2p_4 \rightarrow 1s_4$	2	1	6096.2	16.9	300
$2p_4 \rightarrow 1s_5$	2	2	5944.8	10.5	500
$2p_4 \rightarrow \Sigma 1s$			Red-orange	51.2	

Other transitions	ΣA		λ	$A(\times 10^6)$
$2p_1 \rightarrow \Sigma 1s$	87.9	$2s_2 - 2p_1$	1.5231 μm	0.802
$2p_2 \rightarrow \Sigma 1s$	116.6	$2p_2$	1.1767 μm	4.089
$2p_3 \rightarrow \Sigma 1s$	61.7	$2p_3$	1.1602 μm	0.801 (first gas laser)
$2p_4 \rightarrow \Sigma 1s$	51.7	$2p_4$	1.1523 μm	6.537 (first gas laser)
$2p_5 \rightarrow \Sigma 1s$	53.3	$2p_5$	1.1409 μm	2.301 (first gas laser)
$2p_6 \rightarrow \Sigma 1s$	53.6	$2p_6$	1.0844 μm	7.543
$2p_7 \rightarrow \Sigma 1s$	49.3	$2p_7$	1.0621 μm	0.816
$2p_8 \rightarrow \Sigma 1s$	41.2	$2p_8$	1.0295 μm	0.726
$2p_9 \rightarrow \Sigma 1s$	43.3	$2p_9$	Forbidden	—
$2p_{10} \rightarrow \Sigma 1s$	33.6	$2p_{10}$	0.8895 μm	1.708