Chapters ~~4~~ 3 and ~~5~~ 4
Discrete Time Fourier Transform

Recall that we wrote the sampled signal $x_s(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT)$. We calculate its Fourier Transform.

We do the following:

Ex. Find the Continuous Time Fourier Transform of $\delta(t - kT)$.

Ex. Using superposition, find the CT Fourier Transform of $x_s(t)$.

Now, you just calculated that

$$x_s(t) \leftrightarrow \sum_{n=-\infty}^{\infty} x(nT)e^{-jn\omega T}$$

Let $x(nT) = x[n]$ and make a change of variables $\Omega = \omega T$ (we'll talk more about this later — it relates the discrete-frequency variable Ω to the continuous frequency variable ω via the sampling period T) and we get:

$$\text{DTFT} : X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

- Discrete in time but continuous in frequency and periodic
- Spectrum of discrete signal $x[n]$
- Will compare the DTFT of a discrete signal $x[n]$ with the Continuous Time Fourier Transform of a sampled continuous time signal $x_s(t) = x(t)p(t)$

Formula to calculate inverse DTFT (since $X(\Omega)$ is periodic):

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega)e^{j\Omega n} d\Omega$$

where DTFT is periodic in frequency with period 2π. Why? Because $e^{j\Omega}$ is periodic with period 2π.

$$e^{j\Omega} = e^{j(\Omega+2\pi)} = e^{j\Omega}e^{j2\pi} = e^{j\Omega}.$$

Not all DTFTs converge due to the infinite sum.

DTFT Theorems

We cover a few here and you can read about the rest in the textbook.

LINEARITY

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(\Omega) + bX_2(\Omega)$$

TIME SHIFT

$$x[n] \longleftrightarrow X(\Omega)$$

$$x[n - n_0] \longleftrightarrow ?$$

$$x[n - n_0] \longleftrightarrow e^{-j\Omega n_0} X(\Omega)$$

So a shift in time causes a linear phase shift in frequency – adds a linear term to the phase of the DTFT.

Frequency Shift

$$x[n] \longleftrightarrow X(\Omega)$$

$$e^{j\Omega_0 n}x[n] \longleftrightarrow ?$$

Modulation causes a shift in frequency.

CONVOLUTION IN TIME

As usual,

$$x_1[n] * x_2[n] \longleftrightarrow X_1(\Omega)X_2(\Omega)$$

Ex. Given $h[n] = a^n u[n], |a| < 1$. Find its inverse system $h_i[n]$.

Modulation– MULTIPLICATION OF SIGNALS

$$x_1[n]x_2[n] \longleftrightarrow \frac{1}{2\pi} X_1(\Omega) \bigotimes X_2(\Omega)$$

where $\bigotimes$ denotes CIRCULAR CONVOLUTION:

$$X_1(\Omega) \bigotimes X_2(\Omega) = \int_{2\pi} X_1(\theta) X_2(\Omega - \theta)) d\theta$$

$$y[n] = x_1[n]x_2[n] \longleftrightarrow \text{Take its DTFT :}$$

$$Y(\Omega) = \sum_n y[n]e^{-j\Omega n} = \sum_n x_1[n]x_2[n]e^{-j\Omega n}$$

$$x_1[n] = \frac{1}{2\pi} \int_{2\pi} X_1(a) e^{jan} da$$

$$x_2[n] = \frac{1}{2\pi} \int_{2\pi} X_2(b) e^{jbn} db$$

$$Y(\Omega) = \sum_n \left[\frac{1}{2\pi} \int_{da} X_1(a) e^{jan} da \right] \left[\frac{1}{2\pi} \int_{db} X_2(b) e^{jbn} db \right] e^{-j\Omega n}$$

$$= \left(\frac{1}{2\pi} \right)^2 \int_{da} \int_{db} X_1(a) X_2(b) \sum_n e^{j(a+b)n} e^{-j\Omega n} da db$$

Now,

$$\sum_n e^{j(a+b)n} e^{-j\Omega n}$$

is just the DTFT of

$$e^{j(a+b)n}$$

that is,

$$e^{j(a+b)n} \leftrightarrow 2\pi\delta(\Omega - a - b)$$

So,

$$\begin{aligned} Y(\Omega) &= \frac{1}{2\pi} \int_{\frac{da}{2\pi}} \int_{\frac{db}{2\pi}} X_1(a) X_2(b) \delta(\Omega - a - b) da db \\ &= \frac{1}{2\pi} \int_{2\pi} X_1(a) X_2(\Omega - a) da = \frac{1}{2\pi} X_1(\Omega) \bigotimes X_2(\Omega) \end{aligned}$$

Ex. 1 Find $X(\Omega)$ where $x[n] = a^n u[n]$, $|a| < 1$. What if $|a| > 1$?

Ex. 2 $y[n] = a^n u[-n]$, $|a| > 1$. Find $Y(\Omega)$.
What if $|a| < 1$?

Ex. 3 Rectangular pulse, $p[n] = u[n] - u[n - N]$. Find $P(\Omega)$.
Show that this filter has a *linear phase* term.

Ex. Find $H(\Omega)$ for

$$h[n] = \delta[n] + 2\delta[n-1] + 2\delta[n-2] + \delta[n-3]$$

and show that the filter has a linear phase term.

Ex. $x[n] = (.9)^{|n|}$. Find its DTFT.

Relationship between DTFT and Z-Transform

We already saw the DTFT as the Z-transform of $x[n]$ evaluated on the unit circle when we discussed the frequency response:

$$X(\Omega) = X(z)|_{z=e^{j\Omega}}$$

For a nice illustration of the above see figure 4.2.9 of your textbook, on page 260.

If the ROC for the Z-transform contains the unit circle, we can get DTFT from the Z-transform by substitution (compare the DTFT of $a^n u[n]$ with its Z-transform). Using this, we can extend our filtering results provided in Z domain to frequency domain.

We'll see that the DTFT exists in cases where the ROC of the Z-transform does not include the unit circle (e.g. for periodic discrete-time signals) – analogous to the CT Fourier Transform and Laplace Transform.

The Frequency Response of an LTI Discrete Time System

Consider an LTI system with impulse response $h[n]$. The DTFT of $h[n]$ is denoted by $H(\Omega)$ and is given by

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$$

On the other hand, let us calculate the response to a complex exponential signal $x[n] = e^{j\Omega n}$ via the following convolution:

$$\begin{aligned} y[n] &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]e^{j\Omega(n-k)} \\ &= e^{j\Omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k} \\ &= e^{j\Omega n} H(\Omega) \qquad (1) \end{aligned}$$

For this reason, we call $H(\Omega)$, the *Frequency Response* of the system (i.e. response to an excitation at frequency Ω).

In genral, we can use the convolution/multiplication property of DTFT (see page 4) to get:

$$Y(\Omega) = X(\Omega)H(\Omega)$$

or

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} \qquad (2)$$

As seen before (when we discussed Z-transforms):

$$H(\Omega) = |H(\Omega)|e^{j\theta(\Omega)}$$

where $|H(\Omega)|$ is called the *magnitude response*, while $\theta(\Omega)$ is called *phase response*.

Now we use equation (1) on page 11 to specify the response to a sinusoidal:
Assume $x[n] = A\cos(\Omega_0 n + \phi)$. Using Euler's formula and equation (1), we have:

$$y[n] = v[n] + v^*[n]$$

$$y[n] = \frac{A}{2}\left\{e^{j\phi}|H(\Omega_0)|e^{j\theta(\Omega_0)}e^{j\Omega_0 n} + e^{-j\phi}|H(\Omega_0)|e^{-j\theta(\Omega_0)}e^{-j\Omega_0 n}\right\}$$

or

$$y[n] = A|H(\Omega)|\cos\left(\Omega_0 n + \theta(\Omega_0) + \phi\right)$$

Note: Above, we assume that the input has been present for all time prior to time instant n. This means that the system is in steady state (this is related to the particular solution y_p). In practice, the excitation to a system is usually a right-sided signal. Hence the response to such an input also includes a transient component. This involves calculation a transient response (we called this y_c or complementary solution). For causal stable systems, this transient response can be shown to decay to zero as time passes. In filter design, we consider causal stable systems, hence focusing mostly on steady state response.

Example: Write down the generic difference equation for an IIR filter. Using the above formula, write down the frequency response of a generic IIR filter.

Example: Compute the frequency response of a moving average filter. Draw the graphs for magnitude and phase responses of the moving average filter of length 5.

Concept of Filtering

Consider an LTI system whose magnitude response has the following characteristics:

$$|H(\Omega)| \approx \begin{cases} 0 & 0 \le |\Omega| \le \Omega_c \\ 1 & \Omega_c \le \Omega \le \pi \end{cases}$$

What is the output, if we input $x[n] = A\cos(\Omega_1 n) + B\cos(\Omega_2 n)$?

Why is this called a high pass (HP) filter?

Example: Design a FIT filter of length 3, which passes angular frequency of $\Omega = 0.4$ but rejects angular frequency of $\Omega = 0.1$.

Discrete time Fourier Transform of Periodic Sequences

Here we study the DTFT of periodic sequences. We'll start by looking at the Fourier Series expansion, analogous to what we did in continuous time. Then we will derive the same result using a different approach that will lead us into the Discrete Fourier Transform for finite length sequences.

Recall that for **continuous time periodic signals**, we found the Fourier transform by first doing a Fourier series expansion

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{synthesis equation} \tag{3}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \qquad \text{analysis equation} \tag{4}$$

then using the fact that a complex exponential in time transforms to an impulse in the frequency domain

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - \omega_0)$$

and linearity of the Fourier transform, we get that the CTFT of a periodic signal is made up of harmonically-related impulses with area $2\pi a_k$

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

When a periodic discrete signal is clearly derived (sampled) from a periodic continuous-time signal, the sampling theorems can be invoked to calculate the DTFT.

Example What is the DTFT of $e^{j\Omega_0 n}$?

$$e^{j\Omega_0 n} \longleftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \delta(\Omega - \Omega_0 + 2\pi l)$$

Ex. Find the DTFT of the discrete-time impulse train

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN].$$

We see that:

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN] \leftrightarrow \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi k}{N}) = P(\Omega)$$

Ex. Fourier Series analysis by inspection. Find and sketch a_k for

$$x[n] = 2 + 2\cos(\frac{\pi}{2}n) + \cos(\frac{\pi}{3}n)$$

Here we outline systematic approach to *all* periodic discrete signals (including those that are not constructed by sampling a periodic continuous time signal).

Discrete-time Fourier Series and Transforms for Periodic Signals

Periodic signals can also be described by a Fourier Series expansion:

$$x[n] = \sum_{k \in <N>} a_k e^{jk\Omega_0 n} \qquad \text{synthesis equation} \qquad (5)$$

$$a_k = \frac{1}{N} \sum_{n \in <N>} x[n] e^{-jk\Omega_0 n} \qquad \text{analysis equation} \qquad (6)$$

As one would expect, the integral in time goes to a sum. However, there is one more key difference: *the sum in the synthesis equation is finite!* (over an interval the length of one period).

In the next few pages, we will see

$$a_k = \frac{1}{N} X_0(\frac{2\pi k}{N})$$

where $X_0(\Omega) = \sum_{n=-\infty}^{\infty} x_0[n] e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\Omega n}$

So, we have expressed periodic $x[n]$ as a finite sum of complex exponentials with discrete frequencies $\frac{2\pi k}{N}$. This means that we have

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - \frac{2\pi}{N})$$

Now, recall that $\Omega_0 = \frac{2\pi}{N}$. Since $e^{jk\Omega_0 n} = e^{\frac{jk2\pi n}{N}} = e^{\frac{j(k+N)2\pi n}{N}}$ (since $e^{j2\pi n} = 1, \forall n$), the a_k's are periodic with period N and only N terms are needed in the sum. This means that $a_{k+N} = a_k$, and hence *a periodic discrete time signal is represented by N values in time and N values in frequency!*

Now let's derive equations taking a different approach that will lead to different insights.

Notation: $x[n]$ is a periodic signal with period N. Let $x_0[n]$ be the part of $x[n]$ that is repeated, i.e.

$$x_0[n] = \begin{cases} x[n], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise.} \end{cases}$$

We can take the DTFT of $x_0[n]$:

$$X_0(\Omega) = \sum_{n=-\infty}^{\infty} x_0[n]e^{-jn\Omega} = \sum_{n=0}^{N-1} x_0[n]e^{-jn\Omega}$$

Now, we can also write $x[n]$ as an infinite sum of the function $x_0[n]$ shifted N units at a time:

$$x[n] = \sum_{k=-\infty}^{\infty} x_0[n-kN] = \sum_{k=-\infty}^{\infty} x_0[n] * \delta[n-kN] = x_0[n] * \sum_{k=-\infty}^{\infty} \delta[n-kN]$$

We get from the convolution property that its DTFT $X(\Omega)$ is:

$$x[n] = x_0[n] * p[n] \longleftrightarrow X_0(\Omega)P(\Omega) = X(\Omega)$$

then using the DTFT of the impulse train that we just found

$$\begin{aligned} X(\Omega) &= X_0(\Omega)\left(\frac{2\pi}{N}\sum_k \delta(\Omega - \frac{2\pi k}{N})\right) &(7) \\ &= \frac{2\pi}{N}\sum_k X_0(\frac{2\pi k}{N})\delta(\Omega - \frac{2\pi k}{N}) &(8) \end{aligned}$$

by the property of multiplication by an impulse.

Ex. Examine $X_0(\frac{2\pi k}{N})$. How many distinct values does it have?

The inverse DTFT formula is:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_0^{2\pi}[\frac{2\pi}{N}\sum_{k=-\infty}^{\infty} X_0(\frac{2\pi k}{N})\delta(\Omega - \frac{2\pi k}{N})]e^{j\Omega n}d\Omega$$

$$= \frac{1}{N}\sum_{k=-\infty}^{\infty} X_0(\frac{2\pi k}{N})\int_0^{2\pi}\delta(\Omega - \frac{2\pi k}{N})e^{j\Omega n}d\Omega = \frac{1}{N}\sum_{k=0}^{N-1} X_0(\frac{2\pi k}{N})e^{\frac{j2\pi kn}{N}}$$

by the sifting property and because only the impulses for k between 0 and $N-1$ occur in the range from 0 to 2π.

Therefore, if we compare to the Fourier Series formulation on page 107, we get that

$$a_k = \frac{1}{N}X_0(\frac{2\pi k}{N})$$

In summary, we have:

$$x[n] = x_0[n] * \sum_{k=-\infty}^{\infty} \delta[n - kN]$$

$$X_0(\Omega) = \sum_{n=0}^{N-1} x_0[n] e^{-j\Omega n}$$

$$X(\Omega) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} X_0(\frac{2\pi k}{N}) \delta(\Omega - \frac{2\pi k}{N})$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_0(\frac{2\pi k}{N}) e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{N-1} a_k e^{\frac{j2\pi kn}{N}}$$

$$a_k = \frac{1}{N} X_0(\frac{2\pi k}{N})$$

The procedure to calculate a DTFT of a periodic DT signal is as follows:

1. Start with $x_0[n]$ and N.
2. Find $X_0(\Omega) = \sum_{n=-\infty}^{\infty} x_0[n] e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\Omega n}$
3. Obtain $X_0(\Omega)$ at $\Omega = \frac{2\pi k}{N}, k = 0, 1, \ldots, N-1$ (this means you need to calculate transformed signal X_0 only at N values.
4. Obtain
$$X(\Omega) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} X_0(\frac{2\pi k}{N}) \delta(\Omega - \frac{2\pi k}{N})$$

We will see that this way of looking at the signal gives us a very nice interpretation of DFT.

Ex. $x[n] = 1$. Find $X(\Omega)$