UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 101 - Fall 2003
Linear Systems Fundamentals

MIDTERM EXAM

Open book, closed notes, no calculators

PRINT YOUR NAME ______________________________

Signature ______________________________

Student ID Number ______________________________

Problem	Weight	Score
1	30 pts	
2	30 pts	
3	30 pts	
4	30 pts	
Total	120 pts	
Percent	100 %	

Please do not begin until told.

Write your name on all pages.

Show your work.

Don’t panic!

1. There are exactly six complex numbers that satisfy the equation

$$z^6 - 1 = 0.$$

These numbers are called the "sixth roots of unity."

(a) [12 pts] Determine the sixth roots of unity, and express them in polar form:

$$z = re^{j\theta}, \text{where } r \geq 0,\ 0 \leq \theta < 2\pi,$$

and in cartesian form:

$$z = a + jb, \text{where } a \text{ and } b \text{ are real.}$$

The condition $z^6 - 1$ implies that $|z^6| = |z|^6 = 1$.
It follows that $|z| = 1$, so z must be of the form $z = e^{j\theta}$, for some $\theta \in \mathbb{R}$.
Specifically, $z^6 = (e^{j\theta})^6 = e^{j6\theta} = 1$ implies that $\theta = 2\pi k/6$, for some $k \in \mathbb{Z}$.
Hence, the sixth roots of unity can be expressed as:

$$\begin{array}{rcccrcl}
z_0 & = & e^{j0} & = & 1 & + & j0 \\
z_1 & = & e^{j\frac{2\pi}{6}} = e^{j\frac{\pi}{3}} & = & \frac{1}{2} & + & j\frac{\sqrt{3}}{2} \\
z_2 & = & e^{j\frac{4\pi}{6}} = e^{j\frac{2\pi}{3}} & = & -\frac{1}{2} & + & j\frac{\sqrt{3}}{2} \\
z_3 & = & e^{j\frac{6\pi}{6}} = e^{j\pi} & = & -1 & + & j0 \\
z_4 & = & e^{j\frac{8\pi}{6}} = e^{j\frac{4\pi}{3}} & = & -\frac{1}{2} & - & j\frac{\sqrt{3}}{2} \\
z_5 & = & e^{j\frac{10\pi}{6}} = e^{j\frac{5\pi}{3}} & = & \frac{1}{2} & - & j\frac{\sqrt{3}}{2}
\end{array}$$

1. (b) [6 pts] Mark the locations of the six numbers from part (a) in the complex plane shown in the figure below. (The unit circle is included in the figure for reference purposes.)

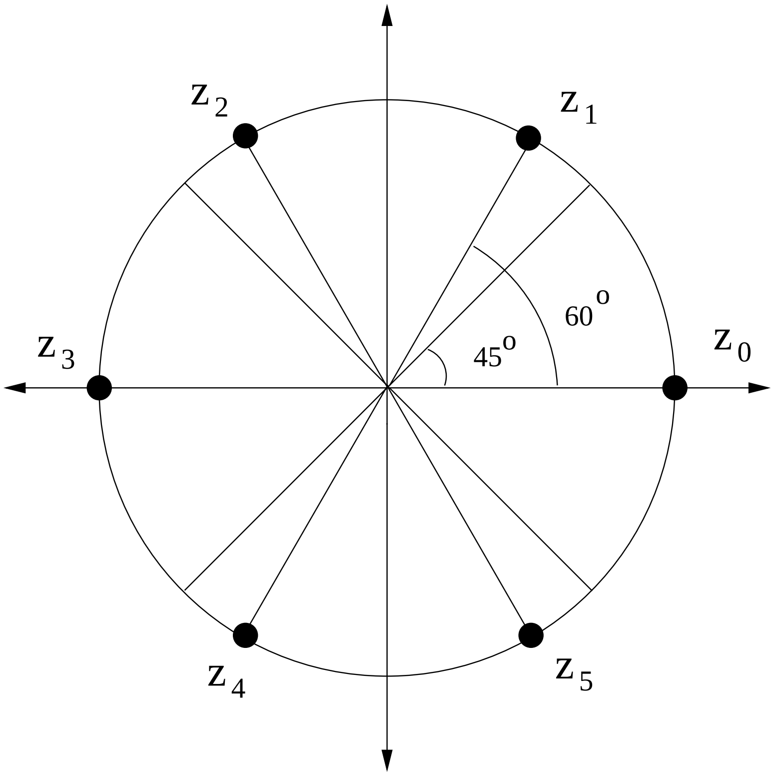

Figure 1: Complex plane with unit circle

1. (c) [12 pts] The polynomial $z^6 - 1$ has the following factorization:

$$\begin{aligned} z^6 - 1 &= (z^3 - 1)(z^3 + 1) && (1) \\ &= (z-1)(z^2+z+1)(z+1)(z^2-z+1) && (2) \\ & && (3) \end{aligned}$$

Determine which of the sixth roots of unity satisfy each equation below. (Note: Each of the roots of unity will satisfy exactly one of the equations.)

(i) $z - 1 = 0$: z_0

(ii) $z^2 + z + 1 = 0$: $\{z_2, z_4\}$

These solutions can be found by trial and error using complex arithmetic.
Alternatively, you can use the quadratic formula (remember that?), which states that the solutions to the real quadratic equation:

$$az^2 + bz + c = 0$$

are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

(iii) $z + 1 = 0$: z_3

(iv) $z^2 - z + 1 = 0$: $\{z_1, z_5\}$

2. Consider the following three systems, denoted A, B, and C:

   A: An input $x(t)$ to this continuous-time system produces the output

   $$y(t) = x(3t + 2).$$

   B: An input $x[n]$ to this discrete-time system produces the output

   $$y[n] = x[n] + 2.$$

   C: This continuous-time, linear time-invariant (LTI) system has impulse response

   $$h(t) = \cos(t)u(t).$$

   Now consider the three system properties:

   - Causality
   - Linearity
   - Stability

   Each of the three systems satisfies a distinct pair of these properties. For each pair of properties, determine the corresponding system, and justify your choice.

   (a)[10 pts] Causal and linear: System **<u>C</u>**

   Causal: $h(t) = 0$, **for** $t < 0$.
   Linear: **from the definition of the system**
   Not stable: $h(t)$ **is not absolutely integrable.**

   (b)[10 pts] Causal and stable: System **<u>B</u>**

   Causal: **actually, memoryless, because** $y[n]$ **depends only on** $x[n]$.
   Stable: $|x[n]| \leq B$ **for all** n **implies** $|y[n]| \leq B + 2$ **for all** n.
   Not linear: $x[n] = 0$ **for all** n **implies** $y[n] = 2$ **for all** n.
   (But it is "incrementally linear.")

   (c)[10 pts] Stable and linear: System **<u>A</u>**

   Stable: $|x(t)| \leq B$ **for all** t **implies** $|y(t)| = |x(3t + 2)| \leq B$ **for all** t.
   Linear: **scaling property -** $ax(t) \rightarrow ax(3t + 2) = ay(t)$, **and**
   additivity - $x_1(t) + x_2(t) \rightarrow x_1(3t + 2) + x_2(3t + 2) = y_1(t) + y_2(t)$.
   Not causal: **y(0)=x(2), so** $y(t)$ **depends on a future input** $x(2)$.

3. Consider the discrete-time linear time-invariant (LTI) system with **step response**

$$s[n] = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n = 0 \\ 2 & \text{for } n = 1 \\ 1 & \text{for } n = 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

(a) [2 pts] Sketch $s[n]$ precisely.

Left to the reader...

3. (b) [8 pts] Determine the unit impulse response $h[n]$.

Recall that:

$$\delta[n] = u[n] - u[n-1],$$

where $u[n]$ **is the unit step signal, so**

$$h[n] = s[n] - s[n-1].$$

Thus,

$$\begin{array}{rclcr} h[n] & = & 0, & & n < 0, \\ h[0] & = & s[0] - s[-1] & = & 1, \\ h[1] & = & s[1] - s[0] & = & 1, \\ h[2] & = & s[2] - s[1] & = & -1, \\ h[3] & = & s[3] - s[2] & = & -1, \\ h[n] & = & 0 & & n > 3. \end{array}$$

3. (c) [2 pts] Sketch $h[n]$ precisely.

Left to the reader...

3. (d) [8 pts] If the input to the system is $x[n] = \delta[n]+\delta[n-2]$, determine the output $y[n]$.

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= (\delta[n] + \delta[n-2]) * h[n] \\ &= (\delta[n] * h[n]) + (\delta[n-2] * h[n]) \\ &= h[n] + h[n-2] \end{aligned}$$

Thus,

$$\begin{array}{rcccr} y[n] & = & 0, & & n < 0, \\ y[0] & = & h[0] + h[-2] & = & 1, \\ y[1] & = & h[1] + h[-1] & = & 1, \\ y[2] & = & h[2] + h[0] & = & 0, \\ y[3] & = & h[3] + h[1] & = & 0, \\ y[4] & = & h[4] + h[2] & = & -1, \\ y[5] & = & h[5] + h[3] & = & -1, \\ y[n] & = & 0 & & n > 5. \end{array}$$

3. (e) [2 pts] Sketch $y[n]$ precisely.

Left to the reader...

3. (f) [8 pts] If the input to the system is $x[n] = \sum_{k=-\infty}^{\infty} \delta[n-2k]$, determine the output $y[n]$.

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \left(\sum_{k=-\infty}^{\infty} \delta[n-2k]\right) * h[n] \\ &= \sum_{k=-\infty}^{\infty} h[n-2k]. \end{aligned}$$

For a fixed n, we have

$$\begin{aligned} h[n-2k] &= 0, \quad \text{for } n-2k < 0, \\ &= 1, \quad \text{for } n-2k = 0, \\ &= 1, \quad \text{for } n-2k = 1, \\ &= -1, \quad \text{for } n-2k = 2, \\ &= -1, \quad \text{for } n-2k = 3, \\ &= 0, \quad \text{for } n-2k > 3. \end{aligned}$$

So, for $n = 2k$, for an integer k, the sum becomes

$$y[n] = h[0] + h[2] = 1 + (-1) = 0.$$

For for $n = 2k+1$, for an integer k, the sum becomes

$$y[n] = h[1] + h[3] = 1 + (-1) = 0.$$

Therefore,

$$y[n] = 0, \text{ for } n \in \mathbb{Z}.$$

4. Consider the continuous-time, periodic signal

$$x(t) = j\sin\left(\frac{3\pi}{4}t\right) + \cos\left(\frac{\pi}{4}t\right) + 1.$$

(a) [5 pts] Find the fundamental frequency of $x(t)$.

Recall the relations that follow from Euler's formula, namely,

$$\begin{aligned} \sin(\theta) &= \tfrac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \\ \cos(\theta) &= \tfrac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \end{aligned}$$

Now use these directly to express $x(t)$ **as a Fourier Series**

$$x(t) = 1 + \frac{1}{2}\left(e^{j\frac{3\pi}{4}t} - e^{-j\frac{3\pi}{4}t}\right) + \frac{1}{2}\left(e^{j\frac{\pi}{4}t} + e^{-j\frac{\pi}{4}t}\right).$$

It is now clear the fundamental frequency ω_0 **satisfies:**

$$\omega_0 = \frac{\pi}{4}.$$

(b) [5 pts] Find the fundamental period of $x(t)$.

The fundamental period T **satisfies**

$$T = \frac{2\pi}{\omega_0},$$

so, from part (a), we conclude that

$$T = \frac{2\pi}{\pi/4} = 8.$$

4. (c) [5 pts] Determine the Fourier series of $x(t)$.

The Fourier Series representation has the form:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{\pi}{4}t}.$$

From the representation of $x(t)$ in part (a), we can read off the Fourier Series coefficients $\{a_k\}$. Specifically,

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= \tfrac{1}{2} \\
a_{-1} &= \tfrac{1}{2} \\
a_2 &= 0 \\
a_{-2} &= 0 \\
a_3 &= \tfrac{1}{2} \\
a_{-3} &= -\tfrac{1}{2} \\
a_k &= 0, \text{otherwise.}
\end{aligned}$$

4. (d) [5 pts] Determine the average value of $x(t)$ in one period.

Recall that the average value is given by the Fourier Series coefficient a_0, that is,

$$a_0 = \frac{1}{T}\int_T x(t)dt.$$

From part (c), we know the value of a_0, namely:

$$a_0 = 1.$$

4. (e) [5 pts] Determine the average power of $x(t)$ in one period.

From Parseval's formula, we know that

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

The left-hand-side is precisely the average power of $x(t)$ in one period. From the answer to part (c), we can easily compute this quantity using the expression on the right-hand-side:

$$\begin{aligned}\textstyle\sum_{k=-\infty}^{\infty} |a_k|^2 &= |a_0|^2 + |a_1|^2 + |a_{-1}|^2 + |a_3|^2 + |a_{-3}|^2 \\ &= 1 + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} \\ &= 2\end{aligned}$$

4\. (f) [5 pts] Suppose the signal $x(t)$ is the input to a linear, time-invariant (LTI) system with frequency response $H(j\omega)$ given by:

$$H(j\omega) = \begin{cases} 0 & \text{for } |\omega| \leq \frac{\pi}{2} \\ 2 & \text{otherwise.} \end{cases}$$

Determine the output $y(t)$.

We know that the complex exponentials are eigenfunctions of LTI systems.

Specifically,

$$e^{j\omega t} \longmapsto H(j\omega)e^{j\omega t}.$$

By linearity, we also know that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{\pi}{4}t} \longmapsto y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\frac{\pi}{4})e^{jk\frac{\pi}{4}t}.$$

With $H(j\omega)$ as defined, the only nonzero terms in the sum for $y(t)$ correspond to the frequencies $\pm\frac{3\pi}{4}$, corresponding to the Fourier Series coefficients a_3 and a_{-3}.

Therefore,

$$\begin{aligned} y(t) &= 2 \cdot \tfrac{1}{2}e^{j\frac{3\pi}{4}t} + 2 \cdot -\tfrac{1}{2}e^{-j\frac{3\pi}{4}t} \\ &= e^{j\frac{3\pi}{4}t} - e^{-j\frac{3\pi}{4}t} \\ &= 2j\sin(\tfrac{3\pi}{4}t). \end{aligned}$$

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