UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 101 - Winter 2007
Linear Systems Fundamentals

MIDTERM EXAM WITH SOLUTIONS

You are allowed one 2-sided sheet of notes.

No books, no other notes, no calculators.

Students caught cheating will receive a course grade of F and disciplinary action will be vigorously pursued.

PRINT YOUR NAME ______________________________

Signature ______________________________

Student ID Number ______________________________

Problem	Weight	Score
1	20 pts	20
2	30 pts	30
3	30 pts	30
4	20 pts	20
Total	100 pts	100

Please do not begin until told.

Write your name on all pages.

Show your work.

Don't panic!!!

Name/Student ID: ______________________________

1. For each problem below, check one or more boxes to correctly complete the statement. Please justify your answers.

   (a) [5 pts] Let $x(t) = sin(\pi t)u(t)$, where $u(t)$ is the unit step signal. Then the odd part of $x(t)$ equals:

   ☐ $2\sin(\pi t)$
   ☐ $-\sin(\pi t)$
   ☐ $\sin(-2\pi t)$
   ☒ $\frac{1}{2}\sin(\pi t)$

   Justification:

$$\begin{aligned} Odd\{x(t)\} &= \frac{x(t) - x(-t)}{2} \\ x(t) - x(-t) &= \sin(\pi t) \\ Odd\{x(t)\} &= \frac{1}{2}\sin(\pi t) \end{aligned}$$

   (b) [5 pts] Let $z = (1 + j\sqrt{3})e^{-j5\pi/6}$. Then z can be written in the form $z = aj$ where:

   ☒ $a = -2$
   ☐ $a = 2$
   ☐ $a = -j$
   ☐ $a = 1$

   Justification:

$$\begin{aligned} z &= (1 + j\sqrt{3})e^{-j5\pi/6} \\ &= 2e^{j\pi/3}e^{-j5\pi/6} \\ &= 2e^{j(\pi/3-5\pi/6)} \\ &= 2e^{j(-\pi/2)} \\ &= -2j. \end{aligned}$$

   So, $a = -2$.

Name/Student ID: ______________________________

1(c) [5 pts] Let $x(t) = 2 + u(t-8) + u(t-3) - u(t+1)$, where $u(t)$ is the unit step signal. Then

$$\int_{-\infty}^{\infty} \delta(t-9)x(t)dt$$

is equal to:

- ☒ 3
- ☐ 1
- ☐ 4
- ☐ 2

Justification:

$$\int_{-\infty}^{\infty} \delta(t-9)x(t)dt = x(9)$$

$$\begin{aligned} x(9) &= 2 + u(1) + u(6) - u(10) \\ &= 2 + 1 + 1 - 1 \\ &= 3. \end{aligned}$$

(d) [5 pts] The discrete-time signal $x[n] = cos(\frac{\pi}{4}n + \frac{\pi}{8})$ is:

- ☐ aperiodic
- ☐ periodic with fundamental period $N = 4$
- ☒ periodic with fundamental period $N = 8$
- ☐ periodic with fundamental period $N = 16$

Justification:

$$\begin{aligned} \cos(\frac{\pi}{4}n + \frac{\pi}{8}) &= \frac{1}{2}[e^{j(\frac{\pi}{4}n+\frac{\pi}{8})} + e^{-j(\frac{\pi}{4}n+\frac{\pi}{8})}] \\ &= \frac{1}{2}e^{j\frac{\pi}{8}}e^{j\frac{\pi}{4}n} + \frac{1}{2}e^{-j\frac{\pi}{8}}e^{-j\frac{\pi}{4}n}. \end{aligned}$$

Each term in the sum is a constant times a periodic discrete-time complex exponential with fundamental frequency $\omega_0 = \frac{\pi}{4} = \frac{2\pi}{8}$. So, the fundamental period is $N = 8$.

Name/Student ID: ______________________________

2. Consider the discrete-time system defined by:

$$y[n] = x[2n-1]$$

Indicate below which properties the system satisfies. Justify your answers on this page. [Each part is worth 5 points.]

True	False	
☒	☐	Linear Let $\tilde{x}[n] = ax[n]$. The corresponding output is $\tilde{y}[n] = \tilde{x}[2n-1] = ax[2n-1] = ay[n]$. So, scalability holds. Input $x_1[n]$ produces output $y_1[n] = x_1[2n-1]$. Input $x_2[n]$ produces output $y_2[n] = x_2[2n-1]$. Input $x[n] = x_1[n] + x_2[n]$ produces output $y[n] = x[2n-1] = x_1[2n-1] + x_2[2n-1] = y_1[n] + y_2[n]$. So, additivity holds, too, proving linearity.
☒	☐	Stable Clearly, $\lvert y[n]\rvert = \lvert x[2n-1]\rvert$. So, if $\lvert x[n]\rvert \leq B$, for all n, then $\lvert y[n]\rvert = \lvert x[2n-1]\rvert \leq B$. So, the system is stable.
☐	☒	Memoryless For all $n \neq 1$, $y[n]$ does not depend only upon $x[n]$. For example, $y[2] = x[3]$. So, the system is not memoryless.

☐ ☒ Invertible

The system is not invertible because we can have two distinct input signals produce the same output signal. For example, the two distinct input signals

$$x_1[n] = \begin{cases} 0 & \text{for } \; n \text{ odd} \\ 1 & \text{for } \; n \text{ even} \end{cases}$$

and

$$x_2[n] = \begin{cases} 0 & \text{for } \; n \text{ odd} \\ 2 & \text{for } \; n \text{ even} \end{cases}$$

both yield the output $y[n] = 0$, for all n.

☐ ☒ Time-invariant

Let $x_1[n] = x[n - n_0]$. The corresponding output is
$y_1[n] = x_1[2n - 1] = x[(2n - 1) - n_0] = x[2n - (1 + n_0)]$.
On the other hand, we have
$y[n - n_0] = x[2(n - n_0) - 1] = x[(2n - 2n_0) - 1] = x[2n - (1 + 2n_0)]$.
Since $y_1[n] \neq y[n - n_0]$, the system is not time-invariant.

☐ ☒ Causal

For all $n \geq 2$, $y[n]$ depends on a future value of the input.
For example, $y[2] = x[3]$. So, the system is not causal.

Name/Student ID: ____

3. Let S be a linear time-invariant continuous-time system with impulse response:

$$h(t) = \begin{cases} 0 & \text{for } t < 0 \\ 2 & \text{for } 0 \leq t \leq T \\ 0 & \text{for } t > T \end{cases}$$

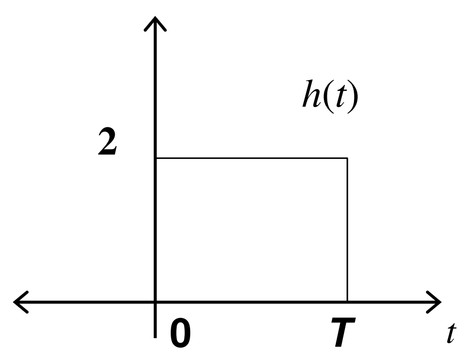
where T is a positive real number.

(a) Let $x_1(t)$ be the continuous-time signal defined by

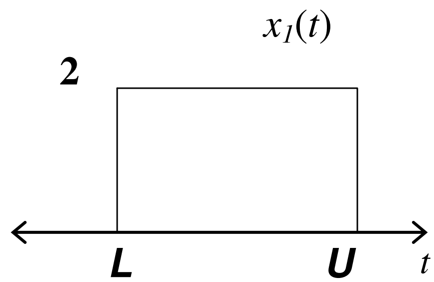
$$x_1(t) = \begin{cases} 0 & \text{for } t < L \\ 2 & \text{for } L \leq t \leq U \\ 0 & \text{for } t > U \end{cases}$$

where L and U are real numbers satisfying $L < U$.

(i) [1 pt] Sketch precisely the impulse response $h(t)$.



(ii) [1 pt] Sketch precisely the signal $x_1(t)$.



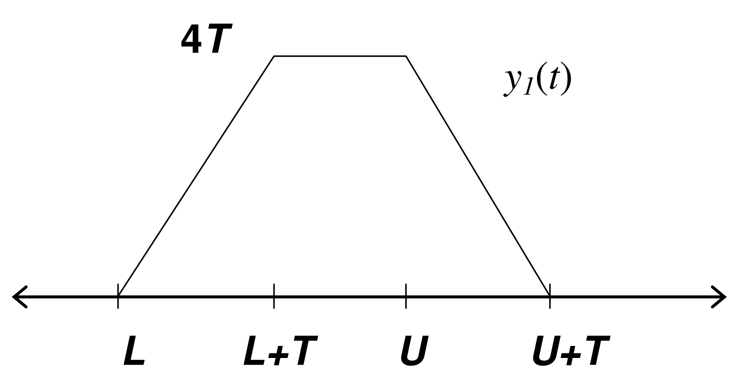
Name/Student ID: ______________________________

3(b) Assume that $0 < T < U - L$. Let $y_1(t)$ be the output of the system corresponding to input $x_1(t)$ from part (a).

(i) [6 pts] Determine and sketch precisely the output $y_1(t)$.
The output is obtained by convolving the input $x_1(t)$ with the impulse response $h(t)$:

$$\begin{aligned} y_1(t) &= x_1(t) * h(t) \\ &= \int_{-\infty}^{\infty} x_1(\tau)h(t-\tau)d\tau. \end{aligned}$$

$t < L$	$y_1(t) = 0;$
$L \leq t < L + T$	$y_1(t) = \int_L^t 4d\tau = 4(t - L);$
$L + T \leq t < U$	$y_1(t) = \int_{t-T}^t 4d\tau = 4T;$
$U \leq t < U + T$	$y_1(t) = \int_{t-T}^U 4d\tau = 4(U + T - t);$
$U + T \leq t$	$y_1(t) = 0.$



(ii) [4 pts] What is the maximum value achieved by $y_1(t)$?

The maximum value is $4T$.

(iii) [4 pts] For which values of t does $y_1(t)$ achieve that maximum value?

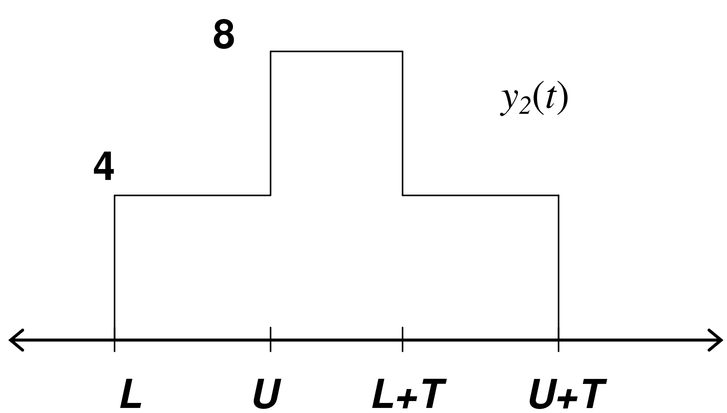
The range of t for which the maximum value is achieved is:

$$L + T \leq t \leq U.$$

3(c) Now assume that $T > U - L$. Let $x_2(t) = 2\delta(t - L) + 2\delta(t - U)$, and let $y_2(t)$ be the output of the system corresponding to the input $x_2(t)$.

(i) [6 pts] Determine and sketch precisely the output $y_2(t)$. The output is obtained by convolving the input $x_2(t)$ with the impulse response $h(t)$:

$$\begin{aligned} y_2(t) &= x_1(t) * h(t) \\ &= 2h(t - L) + 2h(t - U). \end{aligned}$$



Name/Student ID: ______________________________

(ii) [4 pts] What is the maximum value achieved by $y_2(t)$?

The maximum value is 8.

(iii) [4 pts] For which values of t does $y_2(t)$ achieve that maximum value?

The range of t for which the maximum value is achieved is:

$$U \leq t \leq L + T.$$

Name/Student ID: ____________________

4. Consider the continuous-time signal $x(t) = -1 + j\cos(\frac{\pi}{3}t) - \sin(\frac{\pi}{2}t)$.

   (a) [2 pts] Find the fundamental frequency of $x(t)$.

$$x(t) = -1 + j\left[\frac{e^{j\frac{\pi}{3}t} + e^{-j\frac{\pi}{3}t}}{2}\right] - \left[\frac{e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{2}t}}{2j}\right].$$

   Note that $\frac{\pi}{3} = \frac{2\pi}{6}$ and $\frac{\pi}{2} = \frac{3\pi}{6}$. So, the fundamental frequency is:

$$\omega_0 = \frac{\pi}{6}.$$

   (b) [2 pts] Find the fundamental period of $x(t)$.

$$T = \frac{2\pi}{\omega_0} = 12.$$

   (c) [4 pts] Determine the Fourier series coefficients $\{a_k\}$ of $x(t)$.
   Reading the coefficients from the expression in part (a), we get:

$$\begin{aligned} a_0 &= -1 \\ a_2 &= j/2 \\ a_{-2} &= j/2 \\ a_3 &= -1/2j = j/2 \\ a_{-3} &= 1/2j = -j/2 \\ a_k &= 0, \text{ otherwise.} \end{aligned}$$

Name/Student ID: ______________________________

(d) [1 pt] What is the average value of $x(t)$ in one period.

The average value is:

$$a_0 = \frac{1}{T}\int_T x(t)dt = -1,$$

from part (c).

(e) [1 pt] Determine the average power of $x(t)$.

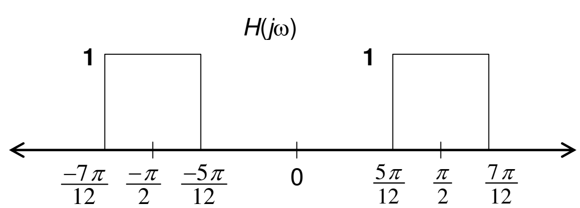
From Parsevals Theorem, the average power P_{ave} is:

$$\begin{aligned} P_{ave} &= \sum_{k=-\infty}^{\infty} |a_k|^2 \\ &= (-1)^2 + |\frac{j}{2}|^2 + |\frac{j}{2}|^2 + |\frac{j}{2}|^2 + |-\frac{j}{2}|^2 \\ &= 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \\ &= 2 \end{aligned}$$

(f) Suppose the signal $x(t)$ is the input to a linear, time-invariant (LTI) system with frequency response $H(j\omega)$ given by:

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega - \frac{\pi}{2}| \leq \frac{\pi}{12} \\ 1 & \text{for } |\omega + \frac{\pi}{2}| \leq \frac{\pi}{12} \\ 0 & \text{otherwise.} \end{cases}$$

(i) [2 pts] Sketch precisely the frequency response $H(j\omega)$.



Name/Student ID: ______________________________

(ii) [2 pts] Determine the fundamental frequency of the system output $y(t)$.

The system output $y(t)$ is given by:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t},$$

where $\{a_k\}$ are the Fourier Series coefficients of the input $x(t)$. Using the definition of $H(j\omega)$ given above, we find:

$$y(t) = \frac{j}{2} e^{j\frac{\pi}{2}t} - \frac{j}{2} e^{-j\frac{\pi}{2}t}.$$

So, the fundamental frequency ω_0 of $y(t)$is:

$$\omega_0 = \frac{\pi}{2}.$$

(iii) [2 pts] Determine the fundamental period of $y(t)$.

$$T = \frac{2\pi}{\omega_0} = 4.$$

(iv) [4 pts] Determine the Fourier series coefficients $\{b_k\}$ of $y(t)$.

From the answer to part (ii) above, we immediately see that the Fourier Series coefficients for $y(t)$ are:

$$\begin{aligned} a_1 &= j/2 \\ a_{-1} &= -j/2 \\ a_k &= 0, \text{ otherwise.} \end{aligned}$$