UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 259BN - Winter Quarter 2008
Trellis-Coded Modulation

Problem Set #1 Due Thursday, January 17, 2008

Reading: Text (Schlegel and Perez), Chapters 1, 2, and 4
Text (Biglieri), Chapters 1 and 2
Supplemental Text (McEliece), Chapter 9

1. Recall Shannon's capacity formula for the band-limited, power-limited Gaussian channel:

$$C = W \log_2(1 + SNR)$$

where the signal-to-noise ratio (SNR) is given by $SNR = P_{ave}/N_0W$.

   (a) Draw a detailed plot of C versus W, assuming fixed average signal power P_{ave} and two-sided noise spectral density $\frac{N_0}{2}$. What is the limiting value of capacity as the bandwidth approaches infinity? Interpret your result.

   (b) In class we rewrote the capacity formula in terms of the average SNR per bit E_b/N_0 and then solved for E_b/N_0 in terms of the spectral efficiency C/W. Give the details of the calculation of E_b/N_0 corresponding to the limiting case of zero spectral efficiency, and provide your interpretation of the result.

   (c) Assume that we use coded modulation on the band-limited AWGN channel to signal at rate $R_b = \frac{1}{2T_s}$. What is the spectral efficiency of this scheme, and what is the minimum E_b/N_0 that permits error-free transmission?

   (d) Referring to part (c), how does the required E_b/N_0 change, if at all, under the assumption that binary modulation is used? If there is a change, can you quantify it more precisely (or at least describe it *qualitatively*)?

   (In 1993, Berrou, Glaviex, and Thitimajshima demonstrated a binary, rate 1/2 encoder and an iterative decoding algorithm that, together, achieved a bit-error-rate of $P_b = 10^{-5}$ at $E_b/N_0 = 0.7$ dB. This "turbo code" shook the world! Comparing its performance to the Shannon limits in parts (c) and (d), you now can understand why.)

2. This problem will familiarize you with properties of generator matrices and physical encoder realizations for convolutional codes. Consider the following polynomial generator matrices for two convolutional codes (assume modulo-2 arithmetic):

$$G_1(D) = \begin{bmatrix} 1+D^2 & D \end{bmatrix}$$

and

$$G_2(D) = \begin{bmatrix} D & 1 & 0 \\ 1 & D^2 & D \end{bmatrix}.$$

(a) For both generator matrices, write the discrete-time input-output relationship, expressing the output bits c_i^j at time j in terms of the appropriate input bits.

(b) For both generator matrices, draw a feed-forward shift-register encoder realization.

(c) Draw the state-diagram and trellis representation of the encoder for $G_1(D)$, and only the trellis representation for $G_2(D)$. (To simplify the drawing, you can use octal notation to represent the states and branch labels in the trellis for $G_2(D)$.)

(d) For both codes, derive a systematic generator matrix, and draw the systematic encoder realization using a shift-register, with feedback as necessary.

(e) Write down a parity-check matrix for each code, based upon your result for part (d).

(f) Determine the minimum Hamming distance of the code generated by $G_1(D)$, and give a code sequence that has minimum Hamming weight.

3. This problem looks at encoders that generate the same convolutional code, and addresses the concept of catastrophic encoders and generator matrices.

(a) The following rate 2/3 generator matrices (encoders) all generate the same code. Prove this explicitly for generator matrices $G_3(D), G_4(D)$, and G₅(D).

$$G_1(D) = \begin{bmatrix} 1 & 1+D+D^2 & 1+D \\ D & 1+D+D^2 & 1 \end{bmatrix}$$

$$G_2(D) = \begin{bmatrix} 1 & 1+D+D^2 & 1+D \\ 0 & 1+D & 1 \end{bmatrix}$$

$$G_3(D) = \begin{bmatrix} 1 & D & 0 \\ 0 & 1+D & 1 \end{bmatrix}$$

$$G_4(D) = \begin{bmatrix} 1+D & 0 & D \\ D & 1+D+D^2 & 1 \end{bmatrix}$$

$$G_5(D) = \begin{bmatrix} 1 & 0 & \frac{D}{1+D} \\ 0 & 1 & \frac{1}{1+D} \end{bmatrix}$$

(b) Determine which of the encoders are catastrophic.

(c) For the catastrophic encoders, give an example of an infinite weight input sequence $u(D)$ that produces a finite weight output sequence $v(D)$.